Capítulo udf

Metatheory of Second-order Logic

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met.1 Introduction

sol:met:int:sec First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, O.s., they give rise to a derivabilidade relation $\vdash$ with the property that for any set of sentenças Γ and sentença φ, $\Gamma \vDash \varphi$ iff $\Gamma \vdash \varphi$. This means in particular that the validities of first-order logic are computavelmente enumerável. There is a computable function $f : \mathbb{N} \to \mathrm{Sent}(\mathcal{L})$ such that the values of f are all and only the valid sentenças of $\mathcal{L}$. This is so because derivações can be enumerated, and those that deriva a single sentença are then mapped to that sentença. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we'll show that second-order logic is not only undecidable, but its validities are not even computavelmente enumerável. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set Γ of sentenças is satisfiable, Γ itself is satisfiable) and the Löwenheim–Skolem Theorem holds for it (if Γ has an infinite model it has a infinito enumerável model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of domínios that first-order logic cannot.

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met.2 Second-order Arithmetic

Recall that the theory **PA** of Peano arithmetic includes the eight axioms of **Q**, sol:met:spa:sec

$$\forall x\, x' \neq \mathsf{o}$$
$$\forall x\, \forall y\, (x' = y' \to x = y)$$
$$\forall x\, (x = \mathsf{o} \lor \exists y\, x = y')$$
$$\forall x\, (x + \mathsf{o}) = x$$
$$\forall x\, \forall y\, (x + y') = (x + y)'$$
$$\forall x\, (x \times \mathsf{o}) = \mathsf{o}$$
$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x)$$
$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (z' + x) = y)$$

plus all sentences of the form

$$(\varphi(\mathsf{o}) \land \forall x\, (\varphi(x) \to \varphi(x'))) \to \forall x\, \varphi(x).$$

The latter is a "schema," O.s., a pattern that generates infinitely many sentenças of the language of arithmetic, one for each fórmula $\varphi(x)$. We call this schema the (first-order) *axiom schema of induction*. In *second-order* Peano arithmetic $\mathbf{PA^2}$, induction can be stated as a single sentence. $\mathbf{PA^2}$ consists of the first eight axioms above plus the (second-order) *induction axiom*:

$$\forall X\, ((X(\mathsf{o}) \land \forall x\, (X(x) \to X(x'))) \to \forall x\, X(x)).$$

It says that if a subset X of the domínio contains $\mathsf{o}^{\mathfrak{M}}$ and with any $x \in |\mathfrak{M}|$ also contains $\prime^{\mathfrak{M}}(x)$ (O.s., it is "closed under successor") it contains everything in the domínio (O.s., $X = |\mathfrak{M}|$).

The induction axiom guarantees that any estrutura satisfying it contains only those membros of $|\mathfrak{M}|$ the axioms require to be there, O.s., the values of $\overline{n}$ for $n \in \mathbb{N}$. A model of $\mathbf{PA^2}$ contains no non-standard numbers.

Teorema met.1. *If $\mathfrak{M} \vDash \mathbf{PA^2}$ then $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$.* *sol:met:spa:thm:sol-pa-standard*

Prova. Let $N = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$, and suppose $\mathfrak{M} \vDash \mathbf{PA^2}$. Of course, for any $n \in \mathbb{N}$, $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \in |\mathfrak{M}|$, so $N \subseteq |\mathfrak{M}|$.

Now for inclusion in the other direction. Consider a variable assignment s with $s(X) = N$. By assumption,

$$\mathfrak{M} \vDash \forall X\, ((X(\mathsf{o}) \land \forall x\, (X(x) \to X(x'))) \to \forall x\, X(x)), \text{ thus}$$
$$\mathfrak{M}, s \vDash (X(\mathsf{o}) \land \forall x\, (X(x) \to X(x'))) \to \forall x\, X(x).$$

Consider the antecedent of this conditional. $\mathrm{Val}^{\mathfrak{M}}(\mathsf{o}) \in N$, and so $\mathfrak{M}, s \vDash X(\mathsf{o})$. The second conjunct, $\forall x\, (X(x) \to X(x'))$ is also satisfied. For suppose $x \in N$. By definition of N, $x = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some n. That gives $\prime^{\mathfrak{M}}(x) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) \in N$. So, $\prime^{\mathfrak{M}}(x) \in N$.

We have that $\mathfrak{M}, s \vDash X(\mathsf{o}) \land \forall x\,(X(x) \rightarrow X(x'))$. Consequently, $\mathfrak{M}, s \vDash \forall x\, X(x)$. But that means that for every $x \in |\mathfrak{M}|$ we have $x \in s(X) = N$. So, $|\mathfrak{M}| \subseteq N$. □

sol:met:spa:
cor:sol-pa-categorical

Corolário met.2. *Any two models of* $\mathbf{PA^2}$ *are isomorphic.*

Prova. By Teorema met.1, the domain of any model of $\mathbf{PA^2}$ is exhausted by $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$. Any such model is also a model of $\mathbf{Q}$. By **??**, any such model is standard, O.s., isomorphic to $\mathfrak{N}$. □

Above we defined $\mathbf{PA^2}$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the *first two* of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

sol:met:spa:
prop:sol-pa-definable

Proposição met.3. *Let* $\mathbf{PA^{2\dagger}}$ *be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then* $\leq$, $+$, *and* $\times$ *are definable in* $\mathbf{PA^{2\dagger}}$.

Prova. To show that $\leq$ is definable, we have to find a formula $\varphi_{\leq}(x, y)$ such that $\mathfrak{N} \vDash \varphi_{\leq}(\overline{n}, \overline{m})$ iff $n \leq m$. Consider the formula

$$\psi(x, Y) \equiv Y(x) \land \forall y\,(Y(y) \rightarrow Y(y'))$$

Clearly, $\psi(\overline{n}, Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m\} \subseteq Y$, so we can take $\varphi_{\leq}(x, y) \equiv \forall Y\,(\psi(x, Y) \rightarrow Y(y))$.

To see that addition is definable observe that $k + l = m$ iff there is a function u such that $u(0) = k$, $u(n') = u(n)'$ for all n, and $m = u(l)$. We can use this equivalence to define addition in $\mathbf{PA^{2\dagger}}$ by the following formula:

$$\varphi_{+}(x, y, z) \equiv \exists u\,(u(\mathsf{o}) = x \land \forall w\, u(x') = u(x)' \land u(y) = z)$$

It should be clear that $\mathfrak{N} \vDash \varphi_{+}(\overline{k}, \overline{l}, \overline{m})$ iff $k + l = m$. □

Problema met.1. Complete the proof of Proposição met.3.

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met.3 Second-order Logic is not Axiomatizable

sol:met:nax:
sec
sol:met:nax:
thm:sol-undecidable

Teorema met.4. *Second-order logic is undecidable.*

Prova. A first-order sentença is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable. □

sol:met:nax:
cor:sol-not-axiomatizable

Teorema met.5. *There is no sound and complete derivação system for second-order logic.*

Prova. Let φ be a sentença in the language of arithmetic. $\mathfrak{N} \vDash \varphi$ iff $\mathbf{PA^2} \vDash \varphi$. Let P be the conjunction of the nine axioms of $\mathbf{PA^2}$. $\mathbf{PA^2} \vDash \varphi$ iff $\vDash P \to \varphi$, O.s., $\mathfrak{M} \vDash P \to \varphi$. Now consider the sentença $\forall z \, \forall u \, \forall u' \, \forall u'' \, \forall L \, (P' \to \varphi')$ resulting by replacing $\mathtt{0}$ by z, $\prime$ by the one-place function variable u, $+$ and $\times$ by the two-place function-variables u' and u'', respectively, and $<$ by the two-place relation variable L and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff $\mathbf{PA^2} \vDash \varphi$ iff $\mathfrak{N} \vDash \varphi$. Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration $f \colon \mathbb{N} \to \mathrm{Sent}(\mathcal{L}_A)$ of the sentenças true in $\mathfrak{N}$. This function would be representable in $\mathbf{Q}$ by some first-order formula $\psi_f(x, y)$. Then the fórmula $\exists x \, \psi_f(x, y)$ would define the set of true first-order sentenças of $\mathfrak{N}$, contradicting Tarski's Theorem. □

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met.4 Second-order Logic is not Compact

explanation Call a set of sentences Γ *finitely satisfiable* if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentenças Γ is finitely satisfiable, it is satisfiable. This property is called *compactness*. It has an equivalent version involving entailment: if $\Gamma \vDash \varphi$, then already $\Gamma_0 \vDash \varphi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \vDash \varphi$, by completeness $\Gamma \vdash \varphi$. But a derivação can only make use of finitely many sentenças of Γ. sol:met:com:sec

Compactness is not true for second-order logic. There are sets of second-order sentenças that are finitely satisfiable but not satisfiable, and that entail some φ without a finite subset entailing φ.

Teorema met.6. *Second-order logic is not compact.* *sol:met:com:thm:sol-undecidable*

Prova. Recall that

$$\mathrm{Inf} \equiv \exists u \, (\forall x \, \forall y \, (u(x) = u(y) \to x = y) \land \exists y \, \forall x \, y \neq u(x))$$

is satisfied in a estrutura iff its domain is infinite. Let $\varphi^{\geq n}$ be a sentence that asserts that the domain has at least n membros, p.ex.,

$$\varphi^{\geq n} \equiv \exists x_1 \, \ldots \exists x_n \, (x_1 \neq x_2 \land x_1 \neq x_3 \land \cdots \land x_{n-1} \neq x_n).$$

Consider the set of sentenças

$$\Gamma = \{\neg \mathrm{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \ldots\}.$$

It is finitely satisfiable, since for any finite subset $\Gamma_0 \subseteq \Gamma$ there is some k so that $\varphi^{\geq k} \in \Gamma$ but no $\varphi^{\geq n} \in \Gamma$ for $n > k$. If $|\mathfrak{M}|$ has k membros, $\mathfrak{M} \vDash \Gamma_0$. But, Γ is not satisfiable: if $\mathfrak{M} \vDash \neg \mathrm{Inf}$, $|\mathfrak{M}|$ must be finite, say, of size k. Then $\mathfrak{M} \nvDash \varphi^{\geq k+1}$. □

Problema met.2. Give an example of a set Γ and a sentença φ so that $\Gamma \vDash \varphi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \nvDash \varphi$.

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met.5 The Löwenheim–Skolem Theorem Fails for Second-order Logic

sol:met:lst: sec The (Downward) Löwenheim–Skolem Theorem states that every set of sentenças with an infinite model has a enumerável model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of sentenças, and that model is enumerável. There is also an Upward Löwenheim–Skolem Theorem, which guarantees that if a set of sentenças has a infinito enumerável model it also has a não enumerável model. Both theorems fail in second-order logic. explanation

sol:met:lst: thm:sol-no-ls **Teorema met.7.** *The Löwenheim–Skolem Theorem fails for second-order logic: There are sentenças with infinite models but no enumerável models.*

Prova. Recall that

$$\text{Count} \equiv \exists z \, \exists u \, \forall X \, ((X(z) \land \forall x \, (X(x) \to X(u(x)))) \to \forall x \, X(x))$$

is true in a estrutura $\mathfrak{M}$ iff $|\mathfrak{M}|$ is enumerável, so $\lnot$Count is true in $\mathfrak{M}$ iff $|\mathfrak{M}|$ is não enumerável. There are such estruturas—take any não enumerável set as the domínio, p.ex., $\wp(\mathbb{N})$ or $\mathbb{R}$. So $\lnot$Count has infinite models but no enumerável models. □

Teorema met.8. *There are sentenças with infinito enumerável but no não enumerável models.*

Prova. Count $\land$ Inf is true in $\mathbb{N}$ but not in any estrutura $\mathfrak{M}$ with $|\mathfrak{M}|$ não enumerável. □

Créditos das Imagens

Referências Bibliográficas