

Capítulo udf

Modal Tableaux

Draft chapter on prefixed tableaux for modal logic. Needs more examples, completeness proofs, and discussion of how one can find countermodels from unsuccessful searches for closed tableaux.

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tab.1 Introduction

nml:tab:int:
sec **Tableaux** are certain (downward-branching) trees of **fórmulas assinadas**, O.s., pairs consisting of a truth value sign ($\mathbb{T}$ or $\mathbb{F}$) and a **sentença**

$$\mathbb{T}\varphi \text{ or } \mathbb{F}\varphi.$$

A **tableau** begins with a number of *assumptions*. Each further **fórmula assinada** is generated by applying one of the inference rules. Some inference rules add one or more **fórmulas assinadas** to a tip of the tree; others add two new tips, resulting in two branches. Rules result in **fórmulas assinadas** where the **fórmula** is less complex than that of the **fórmula assinada** to which it was applied. When a branch contains both $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$, we say the branch is *closed*. If every branch in a **tableau** is closed, the entire **tableau** is closed. A closed **tableau** constitutes a **derivação** that shows that the set of **fórmulas assinadas** which were used to begin the **tableau** are unsatisfiable. This can be used to define a $\vdash$ relation: $\Gamma \vdash \varphi$ iff there is some finite set $\Gamma_0 = \{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ such that there is a closed **tableau** for the assumptions

$$\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}.$$

For modal logics, we have to both extend the notion of **fórmula assinada** and add rules that cover $\Box$ and $\Diamond$. In addition to a sign ($\mathbb{T}$ or $\mathbb{F}$), **fórmulas** in modal **tableaux** also have *prefixes* σ . The prefixes are non-empty sequences of

$\frac{\sigma \mathbb{T} \neg \varphi}{\sigma \mathbb{F} \varphi} \neg \mathbb{T}$	$\frac{\sigma \mathbb{F} \neg \varphi}{\sigma \mathbb{T} \varphi} \neg \mathbb{F}$
$\frac{\sigma \mathbb{T} \varphi \wedge \psi}{\sigma \mathbb{T} \varphi} \wedge \mathbb{T}$ $\sigma \mathbb{T} \psi$	$\frac{\sigma \mathbb{F} \varphi \wedge \psi}{\sigma \mathbb{F} \varphi \mid \sigma \mathbb{F} \psi} \wedge \mathbb{F}$
$\frac{\sigma \mathbb{T} \varphi \vee \psi}{\sigma \mathbb{T} \varphi \mid \sigma \mathbb{T} \psi} \vee \mathbb{T}$	$\frac{\sigma \mathbb{F} \varphi \vee \psi}{\sigma \mathbb{F} \varphi} \vee \mathbb{F}$ $\sigma \mathbb{F} \psi$
$\frac{\sigma \mathbb{T} \varphi \rightarrow \psi}{\sigma \mathbb{F} \varphi \mid \sigma \mathbb{T} \psi} \rightarrow \mathbb{T}$	$\frac{\sigma \mathbb{F} \varphi \rightarrow \psi}{\sigma \mathbb{T} \varphi} \rightarrow \mathbb{F}$ $\sigma \mathbb{F} \psi$

Tabela tab.1: Prefixed **tableau** rules for the propositional connectives

positive integers, O.s., $\sigma \in (\mathbb{Z}^+)^* \setminus \{A\}$. When we write such prefixes without the surrounding $\langle \rangle$, and separate the individual **membros** by .'s instead of ','s. If σ is a prefix, then $\sigma.n$ is $\sigma \frown \langle n \rangle$; p.ex., if $\sigma = 1.2.1$, then $\sigma.3$ is 1.2.1.3. So for instance,

$$1.2 \mathbb{T} \Box \varphi \rightarrow \varphi$$

is a *prefixed fórmula assinada* (or just a *prefixed fórmula* for short).

Intuitively, the prefix names a world in a model that might satisfy the **fórmulas** on a branch of a **tableau**, and if σ names some world, then $\sigma.n$ names a world accessible from (the world named by) σ .

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tab.2 Rules for K

The rules for the regular propositional connectives are the same as for regular propositional signed **tableaux**, just with prefixes added. In each case, the rule applied to a signed **fórmula** $\sigma S \varphi$ produces new **fórmulas** that are also prefixed by σ . This should be intuitively clear: p.ex., if $\varphi \wedge \psi$ is true at (a world named by) σ , then φ and ψ are true at σ (and not at any other world). We collect the propositional rules in **Tabela tab.1**.

The closure condition is the same as for ordinary **tableaux**, although we require that not just the **fórmulas** but also the prefixes must match. So a branch is closed if it contains both

$$\sigma \mathbb{T} \varphi \quad \text{and} \quad \sigma \mathbb{F} \varphi$$

for some prefix σ and **fórmula** φ .

$\frac{\sigma \mathbb{T} \Box \varphi}{\sigma.n \mathbb{T} \varphi} \Box \mathbb{T}$ $\sigma.n$ is used	$\frac{\sigma \mathbb{F} \Box \varphi}{\sigma.n \mathbb{F} \varphi} \Box \mathbb{F}$ $\sigma.n$ is new
$\frac{\sigma \mathbb{T} \Diamond \varphi}{\sigma.n \mathbb{T} \varphi} \Diamond \mathbb{T}$ $\sigma.n$ is new	$\frac{\sigma \mathbb{F} \Diamond \varphi}{\sigma.n \mathbb{F} \varphi} \Diamond \mathbb{F}$ $\sigma.n$ is used

Tabela tab.2: The modal rules for K.

nml:tab:rul:
tab:rules-K

The rules for setting up assumptions is also as for ordinary **tableaux**, except that for assumptions we always use the prefix 1. (It does not matter which prefix we use, as long as it's the same for all assumptions.) So, p.ex., we say that

$$\psi_1, \dots, \psi_n \vdash \varphi$$

iff there is a closed tableau for the assumptions

$$1 \mathbb{T} \psi_1, \dots, 1 \mathbb{T} \psi_n, 1 \mathbb{F} \varphi.$$

For the modal operators $\Box$ and $\Diamond$, the prefix of the conclusion of the rule applied to a **fórmula** with prefix σ is $\sigma.n$. However, which n is allowed depends on whether the sign is $\mathbb{T}$ or $\mathbb{F}$.

The $\Box \mathbb{T}$ rule extends a branch containing $\sigma \mathbb{T} \Box \varphi$ by $\sigma.n \mathbb{T} \varphi$. Similarly, the $\Diamond \mathbb{F}$ rule extends a branch containing $\sigma \mathbb{F} \Diamond \varphi$ by $\sigma.n \mathbb{F} \varphi$. They can only be applied for a prefix $\sigma.n$ which *already* occurs on the branch in which it is applied. Let's call such a prefix “used” (on the branch).

The $\Box \mathbb{F}$ rule extends a branch containing $\sigma \mathbb{F} \Box \varphi$ by $\sigma.n \mathbb{F} \varphi$. Similarly, the $\Diamond \mathbb{T}$ rule extends a branch containing $\sigma \mathbb{T} \Diamond \varphi$ by $\sigma.n \mathbb{T} \varphi$. These rules, however, can only be applied for a prefix $\sigma.n$ which *does not* already occur on the branch in which it is applied. We call such prefixes “new” (to the branch).

The rules are given in **Tabela tab.2**.

The requirement that the restriction that the prefix for $\Box \mathbb{T}$ must be used is necessary as otherwise we would count the following as a closed **tableau**:

1. $1 \mathbb{T} \Box \varphi$ Assumption
 2. $1 \mathbb{F} \Diamond \varphi$ Assumption
 3. $1.1 \mathbb{T} \varphi$ $\Box \mathbb{T} 1$
 4. $1.1 \mathbb{F} \varphi$ $\Diamond \mathbb{F} 2$
- $\otimes$

But $\Box\varphi \not\equiv \Diamond\varphi$, so our proof system would be unsound. Likewise, $\Diamond\varphi \not\equiv \Box\varphi$, but without the restriction that the prefix for $\Box\mathbb{F}$ must be new, this would be a closed tableau:

1.	1 T	$\Diamond\varphi$	Assumption
2.	1 F	$\Box\varphi$	Assumption
3.	1.1 T	φ	$\Diamond\mathbb{T} 1$
4.	1.1 F	φ	$\Box\mathbb{F} 2$
	$\otimes$		

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tab.3 Tableaux for K

Exemplo tab.1. We give a closed tableau that shows $\vdash (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$.

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1.	1 F	$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$	Assumption
2.	1 T	$\Box\varphi \wedge \Box\psi$	$\rightarrow\mathbb{F} 1$
3.	1 F	$\Box(\varphi \wedge \psi)$	$\rightarrow\mathbb{F} 1$
4.	1 T	$\Box\varphi$	$\wedge\mathbb{T} 2$
5.	1 T	$\Box\psi$	$\wedge\mathbb{T} 2$
6.	1.1 F	$\varphi \wedge \psi$	$\Box\mathbb{F} 3$
	$\swarrow \quad \searrow$		
7.	1.1 F	φ	$\wedge\mathbb{F} 6$
8.	1.1 T	φ	$\Box\mathbb{T} 4; \Box\mathbb{T} 5$
	$\otimes$		
	1.1 F	ψ	$\wedge\mathbb{F} 6$
	1.1 T	ψ	$\Box\mathbb{T} 4; \Box\mathbb{T} 5$
	$\otimes$		

Exemplo tab.2. We give a closed tableau that shows $\vdash \Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi)$:

1.	1 F	$\Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi)$	Assumption
2.	1 T	$\Diamond(\varphi \vee \psi)$	$\rightarrow\mathbb{F} 1$
3.	1 F	$\Diamond\varphi \vee \Diamond\psi$	$\rightarrow\mathbb{F} 1$
4.	1 F	$\Diamond\varphi$	$\vee\mathbb{F} 3$
5.	1 F	$\Diamond\psi$	$\vee\mathbb{F} 3$
6.	1.1 T	$\varphi \vee \psi$	$\Diamond\mathbb{T} 2$
	$\swarrow \quad \searrow$		
7.	1.1 T	φ	$\vee\mathbb{T} 6$
8.	1.1 F	φ	$\Diamond\mathbb{F} 4; \Diamond\mathbb{F} 5$
	$\otimes$		
	1.1 T	ψ	$\vee\mathbb{T} 6$
	1.1 F	ψ	$\Diamond\mathbb{F} 4; \Diamond\mathbb{F} 5$
	$\otimes$		

Problema tab.1. Find closed tableaux in K for the following formulas:

1. $\Box\neg p \rightarrow \Box(p \rightarrow q)$

2. $(\Box p \vee \Box q) \rightarrow \Box(p \vee q)$
 3. $\Diamond p \rightarrow \Diamond(p \vee q)$
 4. $\Box(p \wedge q) \rightarrow \Box p$
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tab.4 Soundness for K

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This soundness proof reuses the soundness proof for classical propositional logic, O.s., it proves everything from scratch. That's ok if you want a self-contained soundness proof. If you already have seen soundness for ordinary tableau this will be repetitive. It's planned to make it possible to switch between self-contained version and a version building on the non-modal case.

In order to show that prefixed **tableaux** are sound, we have to show that if

explanation

$$1 \mathbb{T} \psi_1, \dots, 1 \mathbb{T} \psi_n, 1 \mathbb{F} \varphi$$

has a closed **tableau** then $\psi_1, \dots, \psi_n \models \varphi$. It is easier to prove the contrapositive: if for some $\mathfrak{M}$ and world w , $\mathfrak{M}, w \models \psi_i$ for all $i = 1, \dots, n$ but $\mathfrak{M}, w \not\models \varphi$, then no **tableau** can close. Such a countermodel shows that the initial assumptions of the **tableau** are satisfiable. The strategy of the proof is to show that whenever all the prefixed **fórmulas** on a **tableau** branch are satisfiable, any application of a rule results in at least one extended branch that is also satisfiable. Since closed branches are unsatisfiable, any **tableau** for a satisfiable set of prefixed **fórmulas** must have at least one open branch.

In order to apply this strategy in the modal case, we have to extend our definition of “satisfiable” to modal modals and prefixes. With that in hand, however, the proof is straightforward.

Definição tab.3. Let P be some set of prefixes, O.s., $P \subseteq (\mathbb{Z}^+)^* \setminus \{A\}$ and let $\mathfrak{M}$ be a model. A function $f: P \rightarrow W$ is an *interpretation of P in $\mathfrak{M}$* if, whenever σ and $\sigma.n$ are both in P , then $Rf(\sigma)f(\sigma.n)$.

Relative to an interpretation of prefixes P we can define:

1. $\mathfrak{M}$ satisfies $\sigma \mathbb{T} \varphi$ iff $\mathfrak{M}, f(\sigma) \models \varphi$.
2. $\mathfrak{M}$ satisfies $\sigma \mathbb{F} \varphi$ iff $\mathfrak{M}, f(\sigma) \not\models \varphi$.

Definição tab.4. Let Γ be a set of prefixed **fórmulas**, and let $P(\Gamma)$ be the set of prefixes that occur in it. If f is an interpretation of $P(\Gamma)$ in $\mathfrak{M}$, we say that $\mathfrak{M}$ satisfies Γ with respect to f , $\mathfrak{M}, f \models \Gamma$, if $\mathfrak{M}$ satisfies every prefixed **fórmula** in Γ with respect to f . Γ is *satisfiable* iff there is a model $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that $\mathfrak{M}, f \models \Gamma$.

Proposição tab.5. *If Γ contains both $\sigma \mathbb{T} \varphi$ and $\sigma \mathbb{F} \varphi$, for some fórmula φ and prefix σ , then Γ is unsatisfiable.*

Prova. There cannot be a model $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that both $\mathfrak{M}, f(\sigma) \models \varphi$ and $\mathfrak{M}, f(\sigma) \not\models \varphi$. $\square$

Teorema tab.6 (Soundness). *If Γ has a closed tableau, Γ is unsatisfiable.*

[nml:tab:sou:](#)
[thm:tableau-soundness](#)

Prova. We call a branch of a tableau satisfiable iff the set of fórmulas assinadas on it is satisfiable, and let's call a tableau satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable tableau by one of the rules of inference always results in a satisfiable tableau. This will prove the theorem: any closed tableau results by applying rules of inference to the tableau consisting only of assumptions from Γ . So if Γ were satisfiable, any tableau for it would be satisfiable. A closed tableau, however, is clearly not satisfiable, since all its branches are closed and closed branches are unsatisfiable.

Suppose we have a satisfiable tableau, O.s., a tableau with at least one satisfiable branch. Applying a rule of inference either adds fórmulas assinadas to a branch, or splits a branch in two. If the tableau has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended tableau, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let Γ be the set of fórmulas assinadas on that branch, and let $\sigma S \varphi \in \Gamma$ be the fórmula assinada to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, O.s., Γ together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable. First, we consider the possible inferences with only one premise.

1. The branch is expanded by applying $\neg \mathbb{T}$ to $\sigma \mathbb{T} \neg \psi \in \Gamma$. Then the extended branch contains the fórmulas assinadas $\Gamma \cup \{\sigma \mathbb{F} \psi\}$. Suppose $\mathfrak{M}, f \models \Gamma$. In particular, $\mathfrak{M}, f(\sigma) \models \neg \psi$. Thus, $\mathfrak{M}, f(\sigma) \not\models \psi$, O.s., $\mathfrak{M}$ satisfies $\sigma \mathbb{F} \psi$ with respect to f .
2. The branch is expanded by applying $\neg \mathbb{F}$ to $\sigma \mathbb{F} \neg \psi \in \Gamma$: Exercise.
3. The branch is expanded by applying $\wedge \mathbb{T}$ to $\sigma \mathbb{T} \psi \wedge \chi \in \Gamma$, which results in two new fórmulas assinadas on the branch: $\sigma \mathbb{T} \psi$ and $\sigma \mathbb{T} \chi$. Suppose $\mathfrak{M}, f \models \Gamma$, in particular $\mathfrak{M}, f(\sigma) \models \psi \wedge \chi$. Then $\mathfrak{M}, f(\sigma) \models \psi$ and $\mathfrak{M}, f(\sigma) \models \chi$. This means that $\mathfrak{M}$ satisfies both $\sigma \mathbb{T} \psi$ and $\sigma \mathbb{T} \chi$ with respect to f .
4. The branch is expanded by applying $\vee \mathbb{F}$ to $\mathbb{F} \psi \vee \chi \in \Gamma$: Exercise.
5. The branch is expanded by applying $\rightarrow \mathbb{F}$ to $\sigma \mathbb{F} \psi \rightarrow \chi \in \Gamma$: This results in two new fórmulas assinadas on the branch: $\sigma \mathbb{T} \psi$ and $\sigma \mathbb{F} \chi$. Suppose $\mathfrak{M}, f \models \Gamma$, in particular $\mathfrak{M}, f(\sigma) \not\models \psi \rightarrow \chi$. Then $\mathfrak{M}, f(\sigma) \models \psi$ and $\mathfrak{M}, f(\sigma) \not\models \chi$. This means that $\mathfrak{M}, f$ satisfies both $\sigma \mathbb{T} \psi$ and $\sigma \mathbb{F} \chi$.

6. The branch is expanded by applying $\Box\mathbb{T}$ to $\sigma\mathbb{T}\Box\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma.n\mathbb{T}\psi$ on the branch, for some $\sigma.n \in P(\Gamma)$ (since $\sigma.n$ must be used). Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since f is an interpretation of prefixes and both $\sigma, \sigma.n \in P(\Gamma)$, we know that $Rf(\sigma)f(\sigma.n) \Vdash \psi$, O.s., $\mathfrak{M}, f$ satisfies $\sigma.n\mathbb{T}\psi$.
7. The branch is expanded by applying $\Box\mathbb{F}$ to $\sigma\mathbb{F}\Box\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma.n\mathbb{F}\psi$, where $\sigma.n$ is a new prefix on the branch, O.s., $\sigma.n \notin P(\Gamma)$. Since Γ is satisfiable, there is a $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that $\mathfrak{M}, f \models \Gamma$, in particular $\mathfrak{M}, f(\sigma) \not\models \Box\psi$. We have to show that $\Gamma \cup \{\sigma.n\mathbb{F}\psi\}$ is satisfiable. To do this, we define an interpretation of $P(\Gamma) \cup \{\sigma.n\}$ as follows:
 Since $\mathfrak{M}, f(\sigma) \not\models \Box\psi$, there is a $w \in W$ such that $Rf(\sigma)w$ and $\mathfrak{M}, w \not\models \psi$. Let f' be like f , except that $f'(\sigma.n) = w$. Since $f'(\sigma) = f(\sigma)$ and $Rf(\sigma)w$, we have $Rf'(\sigma)f'(\sigma.n)$, so f' is an interpretation of $P(\Gamma) \cup \{\sigma.n\}$. Obviously $\mathfrak{M}, f'(\sigma.n) \not\models \psi$. Since $f(\sigma') = f'(\sigma')$ for all prefixes $\sigma' \in P(\Gamma)$, $\mathfrak{M}, f' \Vdash \Gamma$. So, $\mathfrak{M}, f'$ satisfies $\Gamma \cup \{\sigma.n\mathbb{F}\psi\}$.

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying $\wedge\mathbb{F}$ to $\sigma\mathbb{F}\psi \wedge \chi \in \Gamma$, which results in two branches, a left one continuing through $\sigma\mathbb{F}\psi$ and a right one through $\sigma\mathbb{F}\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \not\models \psi \wedge \chi$. Then $\mathfrak{M}, f(\sigma) \not\models \psi$ or $\mathfrak{M}, f(\sigma) \not\models \chi$. In the former case, $\mathfrak{M}, f$ satisfies $\sigma\mathbb{F}\psi$, O.s., the left branch is satisfiable. In the latter, $\mathfrak{M}, f$ satisfies $\sigma\mathbb{F}\chi$, O.s., the right branch is satisfiable.
2. The branch is expanded by applying $\vee\mathbb{T}$ to $\sigma\mathbb{T}\psi \vee \chi \in \Gamma$: Exercise.
3. The branch is expanded by applying $\rightarrow\mathbb{T}$ to $\sigma\mathbb{T}\psi \rightarrow \chi \in \Gamma$: Exercise. $\square$

Problema tab.2. Complete the proof of **Teorema tab.6**.

nml:tab:sou: **Corolário tab.7.** If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

cor:entailment-soundness

Prova. If $\Gamma \vdash \varphi$ then for some $\psi_1, \dots, \psi_n \in \Gamma$, $\Delta = \{1\mathbb{F}\varphi, 1\mathbb{T}\psi_1, \dots, 1\mathbb{T}\psi_n\}$ has a closed **tableau**. We want to show that $\Gamma \models \varphi$. Suppose not, so for some $\mathfrak{M}$ and w , $\mathfrak{M}, w \Vdash \psi_i$ for $i = 1, \dots, n$, but $\mathfrak{M}, w \not\models \varphi$. Let $f(1) = w$; then f is an interpretation of $P(\Delta)$ into $\mathfrak{M}$, and $\mathfrak{M}$ satisfies Δ with respect to f . But by **Teorema tab.6**, Δ is unsatisfiable since it has a closed **tableau**, a contradiction. So we must have $\Gamma \vdash \varphi$ after all. $\square$

nml:tab:sou: **Corolário tab.8.** If $\vdash \varphi$ then φ is true in all models.

cor:weak-soundness

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$\frac{\sigma \mathsf{T} \Box \varphi}{\sigma \mathsf{T} \varphi} \mathsf{T} \Box$	$\frac{\sigma \mathsf{F} \Diamond \varphi}{\sigma \mathsf{F} \varphi} \mathsf{T} \Diamond$
$\frac{\sigma \mathsf{T} \Box \varphi}{\sigma \mathsf{T} \Diamond \varphi} \mathsf{D} \Box$	$\frac{\sigma \mathsf{F} \Diamond \varphi}{\sigma \mathsf{F} \Box \varphi} \mathsf{D} \Diamond$
$\frac{\sigma.n \mathsf{T} \Box \varphi}{\sigma \mathsf{T} \varphi} \mathsf{B} \Box$	$\frac{\sigma.n \mathsf{F} \Diamond \varphi}{\sigma \mathsf{F} \varphi} \mathsf{B} \Diamond$
$\frac{\sigma \mathsf{T} \Box \varphi}{\sigma.n \mathsf{T} \Box \varphi} 4 \Box$ $\sigma.n$ is used	$\frac{\sigma \mathsf{F} \Diamond \varphi}{\sigma.n \mathsf{F} \Diamond \varphi} 4 \Diamond$ $\sigma.n$ is used
$\frac{\sigma.n \mathsf{T} \Box \varphi}{\sigma \mathsf{T} \Box \varphi} 4r \Box$	$\frac{\sigma.n \mathsf{F} \Diamond \varphi}{\sigma \mathsf{F} \Diamond \varphi} 4r \Diamond$

Tabela tab.3: More modal rules.

Logic	R is ...	Rules
T = KT	reflexive	$\mathsf{T} \Box, \mathsf{T} \Diamond$
D = KD	serial	$\mathsf{D} \Box, \mathsf{D} \Diamond$
K4	transitive	$4 \Box, 4 \Diamond$
B = KTB	reflexive, symmetric	$\mathsf{T} \Box, \mathsf{T} \Diamond$ $\mathsf{B} \Box, \mathsf{B} \Diamond$
S4 = KT4	reflexive, transitive	$\mathsf{T} \Box, \mathsf{T} \Diamond,$ $4 \Box, 4 \Diamond$
S5 = KT4B	reflexive, transitive, euclidean	$\mathsf{T} \Box, \mathsf{T} \Diamond,$ $4 \Box, 4 \Diamond,$ $4r \Box, 4r \Diamond$

nml:tab:mru:
tab:more-rules

Tabela tab.4: **Tableau** rules for various modal logics.

tab.5 Rules for Other Accessibility Relations

In order to deal with logics determined by special accessibility relations, we consider the additional rules in **Tabela tab.3**.

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tab:logics-rules

Adding these rules results in systems that are sound and complete for the logics given in **Tabela tab.4**.

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sec

Exemplo tab.9. We give a closed tableau that shows $\mathbf{S5} \vdash 5$, O.s., $\Box\varphi \rightarrow \Box\Diamond\varphi$.

1.	$1 \mathbb{F} \Box\varphi \rightarrow \Box\Diamond\varphi$	Assumption
2.	$1 \mathbb{T} \Box\varphi$	$\rightarrow \mathbb{F} 1$
3.	$1 \mathbb{F} \Box\Diamond\varphi$	$\rightarrow \mathbb{F} 1$
4.	$1.1 \mathbb{F} \Diamond\varphi$	$\Box \mathbb{F} 3$
5.	$1 \mathbb{F} \Diamond\varphi$	$4r\Diamond 4$
6.	$1.1 \mathbb{F} \varphi$	$\Diamond \mathbb{F} 5$
7.	$1.1 \mathbb{T} \varphi$	$\Box \mathbb{T} 2$
	$\otimes$	

Problema tab.3. Give closed **tableaux** that show the following:

1. $\mathbf{KT5} \vdash B$;
2. $\mathbf{KT5} \vdash 4$;
3. $\mathbf{KDB4} \vdash T$;
4. $\mathbf{KB4} \vdash 5$;
5. $\mathbf{KB5} \vdash 4$;
6. $\mathbf{KT} \vdash D$.

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tab.6 Soundness for Additional Rules

nml:tab:msn: sec We say a rule is sound for a class of models if, whenever a branch in a **tableau** is satisfiable in a model from that class, the branch resulting from applying the rule is also satisfiable in a model from that class.

nml:tab:msn: prop:soundness-T **Proposição tab.10.** $T\Box$ and $T\Diamond$ are sound for reflexive models.

Prova. 1. The branch is expanded by applying $T\Box$ to $\sigma \mathbb{T}\Box\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma \mathbb{T}\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \psi$, O.s., $\mathfrak{M}, f$ satisfies $\sigma \mathbb{T}\psi$.

2. The branch is expanded by applying $T\Diamond$ to $\sigma \mathbb{F}\Diamond\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma \mathbb{F}\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \not\Vdash \psi$, O.s., $\mathfrak{M}, f$ satisfies $\sigma \mathbb{F}\psi$. $\square$

nml:tab:msn: prop:soundness-D **Proposição tab.11.** $D\Box$ and $D\Diamond$ are sound for serial models.

- Prova.* 1. The branch is expanded by applying $D\Box$ to $\sigma\mathbb{T}\Box\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma\mathbb{T}\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \Vdash \psi$, and hence $\mathfrak{M}, f(\sigma) \Vdash \Diamond\psi$. So, $\mathfrak{M}, f$ satisfies $\sigma\mathbb{T}\Diamond\psi$.
2. The branch is expanded by applying $D\Diamond$ to $\sigma\mathbb{F}\Diamond\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma\mathbb{F}\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \not\Vdash \psi$, and hence $\mathfrak{M}, f(\sigma) \not\Vdash \Box\psi$. So, $\mathfrak{M}, f$ satisfies $\sigma\mathbb{F}\Box\psi$. $\square$

Proposição tab.12. $B\Box$ and $B\Diamond$ are sound for symmetric models.

*nml:tab:msn:
prop:soundness-B*

- Prova.* 1. The branch is expanded by applying $B\Box$ to $\sigma.n\mathbb{T}\Box\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma\mathbb{T}\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, $\mathfrak{M}, f(\sigma) \Vdash \psi$. Hence, $\mathfrak{M}, f$ satisfies $\sigma\mathbb{T}\psi$.
2. The branch is expanded by applying $B\Diamond$ to $\sigma.n\mathbb{F}\Diamond\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma\mathbb{F}\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$, $\mathfrak{M}, f(\sigma) \not\Vdash \psi$. Hence, $\mathfrak{M}, f$ satisfies $\sigma\mathbb{F}\psi$. $\square$

Proposição tab.13. $4\Box$ and $4\Diamond$ are sound for transitive models.

*nml:tab:msn:
prop:soundness-4*

- Prova.* 1. The branch is expanded by applying $4\Box$ to $\sigma\mathbb{T}\Box\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma.n\mathbb{T}\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$ and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, and $\mathfrak{M}, f$ satisfies $\sigma.n\mathbb{T}\Box\psi$.
2. The branch is expanded by applying $4\Diamond$ to $\sigma\mathbb{F}\Diamond\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma.n\mathbb{F}\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$ and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$, $\mathfrak{M}, w \not\Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$, and $\mathfrak{M}, f$ satisfies $\sigma.n\mathbb{F}\Diamond\psi$. $\square$

nml:tab:msn:
prop:soundness-4r

Proposição tab.14. $4r\Box$ and $4r\Diamond$ are sound for euclidean models.

- Prova.* 1. The branch is expanded by applying $4r\Box$ to $\sigma.n \mathbb{T}\Box\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma \mathbb{T}\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma).n \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, and $\mathfrak{M}, f$ satisfies $\sigma \mathbb{T}\Box\psi$.
2. The branch is expanded by applying $4r\Diamond$ to $\sigma.n \mathbb{F}\Diamond\psi \in \Gamma$: This results in a new **fórmula assinada** $\sigma \mathbb{T}\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma).n \nVdash \Diamond\psi$, $\mathfrak{M}, w \nVdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$, and $\mathfrak{M}, f$ satisfies $\sigma \mathbb{F}\Diamond\psi$. $\square$

nml:tab:msn:
cor:soundness-logics

Corolário tab.15. The **tableau** systems given in **Tabela tab.4** are sound for the respective classes of models.

-NoValue-

tab.7 Simple Tableaux for S5

nml:tab:s5:
sec

S5 is sound and complete with respect to the class of universal models, O.s., models where every world is accessible from every world. In universal models the accessibility relation doesn't matter: "there is a world w where $\mathfrak{M}, w \Vdash \varphi$ " is true if and only if there is such a w that's accessible from u . So in **S5**, we can define models as simply a set of worlds and a valuation V . This suggests that we should be able to simplify the **tableau** rules as well. In the general case, we take as prefixes sequences of positive integers, so that we can keep track of which such prefixes name worlds which are accessible from others: $\sigma.n$ names a world accessible from σ . But in **S5** any world is accessible from any world, so there is no need to so keep track. Instead, we can use positive integers as prefixes. The simplified rules are given in **Tabela tab.5**.

Exemplo tab.16. We give a simplified closed tableau that shows **S5** $\vdash 5$, O.s., $\Diamond\varphi \rightarrow \Box\Diamond\varphi$.

1.	$1 \mathbb{F} \Diamond\varphi \rightarrow \Box\Diamond\varphi$	Assumption
2.	$1 \mathbb{T} \Diamond\varphi$	$\rightarrow \mathbb{F} 1$
3.	$1 \mathbb{F} \Box\Diamond\varphi$	$\rightarrow \mathbb{F} 1$
4.	$2 \mathbb{F} \Diamond\varphi$	$\Box \mathbb{F} 3$
5.	$3 \mathbb{T} \varphi$	$\Diamond \mathbb{T} 2$
6.	$3 \mathbb{F} \varphi$	$\Diamond \mathbb{F} 4$
	$\otimes$	

$\frac{n \mathbb{T} \Box \varphi}{m \mathbb{T} \varphi} \Box \mathbb{T}$ m is used	$\frac{n \mathbb{F} \Box \varphi}{m \mathbb{F} \varphi} \Box \mathbb{F}$ m is new
$\frac{n \mathbb{T} \Diamond \varphi}{m \mathbb{T} \varphi} \Diamond \mathbb{T}$ m is new	$\frac{n \mathbb{F} \Diamond \varphi}{m \mathbb{F} \varphi} \Diamond \mathbb{F}$ m is used

Tabela tab.5: Simplified rules for **S5**.

nml:tab:s5:
tab:rules-S5

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tab.8 Completeness for K

explanation To show that the method of **tableaux** is complete, we have to show that whenever there is no closed **tableau** to show $\Gamma \vdash \varphi$, then $\Gamma \not\vdash \varphi$. O.s., there is a countermodel. But “there is no closed **tableau**” means that every way we could try to construct one has to fail to close. The trick is to see that if every such way fails to close, then a specific, *systematic and exhaustive* way also fails to close. And this systematic and exhaustive way would close if a closed **tableau** exists. The single tableau will contain, among its open branches, all the information required to define a countermodel. The countermodel given by an open branch in this tableau will contain all the prefixes used on that branch as the worlds, and a **variável proposicional** p is true at σ iff $\sigma \mathbb{T} p$ occurs on the branch.

nml:tab:cpl:
sec

Definição tab.17. A branch in a **tableau** is called complete if, whenever it contains a prefixed **fórmula** $\sigma S \varphi$ to which a rule can be applied, it also contains

1. the prefixed **fórmulas** that are the corresponding conclusions of the rule, in the case of propositional stacking rules;
2. one of the corresponding conclusion **fórmulas** in the case of propositional branching rules;
3. at least one possible conclusion in the case of modal rules that require a new prefix;
4. the corresponding conclusion for every prefix occurring on the branch in the case of modal rules that require a used prefix.

For instance, a complete branch contains $\sigma \mathbb{T} \psi$ and $\sigma \mathbb{T} \chi$ whenever it contains $\mathbb{T} \psi \wedge \chi$. If it contains $\sigma \mathbb{T} \psi \vee \chi$ it contains at least one of $\sigma \mathbb{F} \psi$ and $\sigma \mathbb{T} \chi$. If it contains $\sigma \mathbb{F} \Box$ it also contains $\sigma.n \mathbb{F} \Box$ for at least one n . And whenever it contains $\sigma \mathbb{T} \Box$ it also contains $\sigma.n \mathbb{T} \Box$ for every n such that $\sigma.n$ is used on the branch. explanation

nml:tab:cpl: prop:complete-tableau **Proposição tab.18.** *Every finite Γ has a **tableau** in which every branch is complete.*

Prova. Consider an open branch in a **tableau** for Γ . There are finitely many prefixed **fórmulas** in the branch to which a rule could be applied. In some fixed order (say, top to bottom), for each of these prefixed **fórmulas** for which the conditions (1)–(4) do not already hold, apply the rules that can be applied to it to extend the branch. In some cases this will result in branching; apply the rule at the tip of each resulting branch for all remaining prefixed **fórmulas**. Since the number of prefixed **fórmulas** is finite, and the number of used prefixes on the branch is finite, this procedure eventually results in (possibly many) branches extending the original branch. Apply the procedure to each, and repeat. But by construction, every branch is closed. $\square$

nml:tab:cpl: thm:tableau-completeness **Teorema tab.19 (Completeness).** *If Γ has no closed **tableau**, Γ is satisfiable.*

Prova. By the proposition, Γ has a **tableau** in which every branch is complete. Since it has no closed **tableau**, it has a **tableau** in which at least one branch is open and complete. Let Δ be the set of prefixed **fórmulas** on the branch, and $P(\Delta)$ the set of prefixes occurring in it.

We define a model $\mathfrak{M}(\Delta) = \langle P(\Delta), R, V \rangle$ where the worlds are the prefixes occurring in Δ , the accessibility relation is given by:

$$R\sigma\sigma' \text{ iff } \sigma' = \sigma.n \text{ for some } n$$

and

$$V(p) = \{\sigma : \sigma \mathbb{T} p \in \Delta\}.$$

We show by induction on φ that if $\sigma \mathbb{T} \varphi \in \Delta$ then $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$, and if $\sigma \mathbb{F} \varphi \in \Delta$ then $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

1. $\varphi \equiv p$: If $\sigma \mathbb{T} \varphi \in \Delta$ then $\sigma \in V(p)$ (by definition of V) and so $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.
If $\sigma \mathbb{F} \varphi \in \Delta$ then $\sigma \mathbb{T} \varphi \notin \Delta$, since the branch would otherwise be closed. So $\sigma \notin V(p)$ and thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.
2. $\varphi \equiv \neg\psi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then $\sigma \mathbb{F} \psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ and thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.
If $\sigma \mathbb{F} \varphi \in \Delta$, then $\sigma \mathbb{T} \psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

3. $\varphi \equiv \psi \wedge \chi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then both $\sigma \mathbb{T} \psi \in \Delta$ and $\sigma \mathbb{T} \chi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.
If $\sigma \mathbb{F} \varphi \in \Delta$, then either $\sigma \mathbb{F} \psi \in \Delta$ or $\sigma \mathbb{F} \chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma \nVdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.
4. $\varphi \equiv \psi \vee \chi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then either $\sigma \mathbb{T} \psi \in \Delta$ or $\sigma \mathbb{T} \chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.
If $\sigma \mathbb{F} \varphi \in \Delta$, then both $\sigma \mathbb{F} \psi \in \Delta$ and $\sigma \mathbb{F} \chi \in \Delta$ since the branch is complete. By induction hypothesis, both $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ and $\mathfrak{M}(\Delta), \sigma \nVdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.
5. $\varphi \equiv \psi \rightarrow \chi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then either $\sigma \mathbb{F} \psi \in \Delta$ or $\sigma \mathbb{T} \chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.
If $\sigma \mathbb{F} \varphi \in \Delta$, then both $\sigma \mathbb{T} \psi \in \Delta$ and $\sigma \mathbb{F} \chi \in \Delta$ since the branch is complete. By induction hypothesis, both $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \nVdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.
6. $\varphi \equiv \Box \psi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then, since the branch is complete, $\sigma.n \mathbb{T} \psi \in \Delta$ for every $\sigma.n$ used on the branch, O.s., for every $\sigma' \in P(\Delta)$ such that $R\sigma\sigma'$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma' \Vdash \psi$ for every σ' such that $R\sigma\sigma'$. Therefore, $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.
If $\sigma \mathbb{F} \varphi \in \Delta$, then for some $\sigma.n$, $\sigma.n \mathbb{F} \psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma.n \nVdash \psi$. Since $R\sigma(\sigma.n)$, there is a σ' such that $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.
7. $\varphi \equiv \Diamond \psi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then for some $\sigma.n$, $\sigma.n \mathbb{T} \psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma.n \Vdash \psi$. Since $R\sigma(\sigma.n)$, there is a σ' such that $\mathfrak{M}(\Delta), \sigma' \Vdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.
If $\sigma \mathbb{F} \varphi \in \Delta$, then, since the branch is complete, $\sigma.n \mathbb{F} \psi \in \Delta$ for every $\sigma.n$ used on the branch, O.s., for every $\sigma' \in P(\Delta)$ such that $R\sigma\sigma'$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$ for every σ' such that $R\sigma\sigma'$. Therefore, $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

Since $\Gamma \subseteq \Delta$, $\mathfrak{M}(\Delta) \Vdash \Gamma$. □

Problema tab.4. Complete the proof of [Teorema tab.19](#).

Corolário tab.20. If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

*nml:tab:cpl:
cor:entailment-completeness*

Corolário tab.21. If φ is true in all models, then $\vdash \varphi$.

*nml:tab:cpl:
cor:weak-completeness*

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tab.9 Countermodels from Tableaux

nml:tab:cou:
sec

The proof of the completeness theorem doesn't just show that if $\models \varphi$ then $\vdash \varphi$, [explanation](#) it also gives us a method for constructing countermodels to φ if $\not\models A$. In the case of **K**, this method constitutes a *decision procedure*. For suppose $\not\models \varphi$. Then the proof of [Proposição tab.18](#) gives a method for constructing a complete [tableau](#). The method in fact always terminates. The propositional rules for **K** only add prefixed [fórmulas](#) of lower complexity, O.s., each propositional rule need only be applied once on a branch for any signed formula $\sigma S\varphi$. New prefixes are only generated by the $\Box F$ and $\Diamond T$ rules, and also only have to be applied once (and produce a single new prefix). $\Box T$ and $\Diamond F$ have to be applied potentially multiple times, but only once per prefix, and only finitely many new prefixes are generated. So the construction either results in a closed branch or a complete branch after finitely many stages.

Once a tableau with an open complete branch is constructed, the proof of [Teorema tab.19](#) gives us an explicit model that satisfies the original set of prefixed [fórmulas](#). So not only is it the case that if $\Gamma \models \varphi$, then a closed [tableau](#) exists and $\Gamma \vdash \varphi$, if we look for the closed [tableau](#) in the right way and end up with a “complete” [tableau](#), we'll not only know that $\Gamma \not\models \varphi$ but actually be able to construct a countermodel.

Exemplo tab.22. We know that $\not\models \Box(p \vee q) \rightarrow (\Box p \vee \Box q)$. The construction of a tableau begins with:

1.	$1 F \Box(p \vee q) \rightarrow (\Box p \vee \Box q) \checkmark$	Assumption
2.	$1 T \Box(p \vee q)$	$\rightarrow F 1$
3.	$1 F \Box p \vee \Box q \checkmark$	$\rightarrow F 1$
4.	$1 F \Box p \checkmark$	$\vee F 3$
5.	$1 F \Box q \checkmark$	$\vee F 3$
6.	$1.1 F p \checkmark$	$\Box F 4$
7.	$1.2 F q \checkmark$	$\Box F 5$

The [tableau](#) is of course not finished yet. In the next step, we consider the only line without a checkmark: the prefixed [fórmula](#) $1 T \Box(p \vee q)$ on line 2. The construction of the closed tableau says to apply the $\Box T$ rule for every prefix used on the branch, O.s., for both 1.1 and 1.2:

1.	$1 F \Box(p \vee q) \rightarrow (\Box p \vee \Box q) \checkmark$	Assumption
2.	$1 T \Box(p \vee q)$	$\rightarrow F 1$
3.	$1 F \Box p \vee \Box q \checkmark$	$\rightarrow F 1$
4.	$1 F \Box p \checkmark$	$\vee F 3$
5.	$1 F \Box q \checkmark$	$\vee F 3$
6.	$1.1 F p \checkmark$	$\Box F 4$
7.	$1.2 F q \checkmark$	$\Box F 5$
8.	$1.1 T p \vee q$	$\Box T 2$
9.	$1.2 T p \vee q$	$\Box T 2$

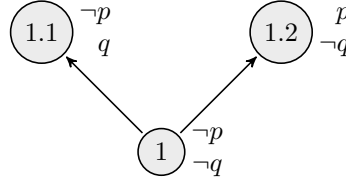


Figura tab.1: A countermodel to $\Box(p \vee q) \rightarrow (\Box p \vee \Box q)$.

Now lines 2, 8, and 9, don't have checkmarks. But no new prefix has been added, so we apply $\vee\mathbb{T}$ to lines 8 and 9, on all resulting branches (as long as they don't close):

nml:tab:cou:
fig:counter-Box

1.	$1 \mathbb{F} \Box(p \vee q) \rightarrow (\Box p \vee \Box q) \checkmark$	Assumption
2.	$1 \mathbb{T} \Box(p \vee q) \checkmark$	$\rightarrow \mathbb{F} 1$
3.	$1 \mathbb{F} \Box p \vee \Box q \checkmark$	$\rightarrow \mathbb{F} 1$
4.	$1 \mathbb{F} \Box p \checkmark$	$\vee \mathbb{F} 3$
5.	$1 \mathbb{F} \Box q \checkmark$	$\vee \mathbb{F} 3$
6.	$1.1 \mathbb{F} p \checkmark$	$\Box \mathbb{F} 4$
7.	$1.2 \mathbb{F} q \checkmark$	$\Box \mathbb{F} 5$
8.	$1.1 \mathbb{T} p \vee q \checkmark$	$\Box \mathbb{T} 2$
9.	$1.2 \mathbb{T} p \vee q \checkmark$	$\Box \mathbb{T} 2$
10.	$1.1 \mathbb{T} p \checkmark$ $\otimes$	$\vee \mathbb{T} 8$
11.	$1.2 \mathbb{T} p \checkmark$ $1.2 \mathbb{T} q \checkmark$ $\otimes$	$\vee \mathbb{T} 9$

There is one remaining open branch, and it is complete. From it we define the model with worlds $W = \{1, 1.1, 1.2\}$ (the only prefixes appearing on the open branch), the accessibility relation $R = \{\langle 1, 1.1 \rangle, \langle 1, 1.2 \rangle\}$, and the assignment $V(p) = \{1.2\}$ (because line 11 contains $1.2 \mathbb{T} p$) and $V(q) = \{1.1\}$ (because line 10 contains $1.1 \mathbb{T} q$). The model is pictured in [Figura tab.1](#), and you can verify that it is a countermodel to $\Box(p \vee q) \rightarrow (\Box p \vee \Box q)$.

Créditos das Imagens

Referências Bibliográficas