

Capítulo udf

Syntax and Semantics

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syn.1 Introduction

nml:syn:int:
sec Modal logic deals with *modal propositions* and the entailment relations among them. Examples of modal propositions are the following:

1. It is necessary that $2 + 2 = 4$.
2. It is necessarily possible that it will rain tomorrow.
3. If it is necessarily possible that φ then it is possible that φ .

Possibility and necessity are not the only modalities: other unary connectives are also classified as modalities, for instance, “it ought to be the case that φ ,” “It will be the case that φ ,” “Dana knows that φ ,” or “Dana believes that φ .”

Modal logic makes its first appearance in Aristotle’s *De Interpretatione*: he was the first to notice that necessity implies possibility, but not vice versa; that possibility and necessity are inter-definable; that If $\varphi \wedge \psi$ is possibly true then φ is possibly true and ψ is possibly true, but not conversely; and that if $\varphi \rightarrow \psi$ is necessary, then if φ is necessary, so is ψ .

The first modern approach to modal logic was the work of C. I. Lewis, culminating with Lewis and Langford, *Symbolic Logic* (1932). Lewis & Langford were unhappy with the representation of implication by means of the material conditional: $\varphi \rightarrow \psi$ is a poor substitute for “ φ implies ψ .” Instead, they proposed to characterize implication as “Necessarily, if φ then ψ ,” symbolized as $\varphi \rightarrow \psi$. In trying to sort out the different properties, Lewis identified five different modal systems, **S1**, . . . , **S4**, **S5**, the last two of which are still in use.

The approach of Lewis and Langford was purely *syntactical*: they identified reasonable axioms and rules and investigated what was provable with those means. A semantic approach remained elusive for a long time, until a first attempt was made by Rudolf Carnap in *Meaning and Necessity* (1947) using the notion of a *state description*, O.s., a collection of atomic sentences (those

that are “true” in that state description). After lifting the truth definition to arbitrary sentences φ , Carnap defines φ to be *necessarily true* if it is true in all state descriptions. Carnap’s approach could not handle *iterated* modalities, in that sentences of the form “Possibly necessarily ... possibly φ ” always reduce to the innermost modality.

The major breakthrough in modal semantics came with Saul Kripke’s article “A Completeness Theorem in Modal Logic” (JSL 1959). Kripke based his work on Leibniz’s idea that a statement is necessarily true if it is true “at all possible worlds.” This idea, though, suffers from the same drawbacks as Carnap’s, in that the truth of statement at a world w (or a state description s) does not depend on w at all. So Kripke assumed that worlds are related by an *accessibility relation* R , and that a statement of the form “Necessarily φ ” is true at a world w if and only if φ is true at all worlds w' *accessible from* w . Semantics that provide some version of this approach are called Kripke semantics and made possible the tumultuous development of modal logics (in the plural).

When interpreted by the Kripke semantics, modal logic shows us what *relational structures* look like “from the inside.” A relational structure is just a set equipped with a binary relation (for instance, the set of students in the class ordered by their social security number is a relational structure). But in fact relational structures come in all sorts of domains: besides relative possibility of states of the world, we can have epistemic states of some agent related by epistemic possibility, or states of a dynamical system with their state transitions, etc. Modal logic can be used to model all of these: the first gives us ordinary, alethic, modal logic; the others give us epistemic logic, dynamic logic, etc.

We focus on one particular angle, known to modal logicians as “correspondence theory.” One of the most significant early discoveries of Kripke’s is that many properties of the accessibility relation R (whether it is transitive, symmetric, etc.) can be characterized *in the modal language* itself by means of appropriate “modal schemas.” Modal logicians say, for instance, that the reflexivity of R “corresponds” to the schema “If necessarily φ , then φ ”. We explore mainly the correspondence theory of a number of classical systems of modal logic (p.ex., **S4** and **S5**) obtained by a combination of the schemas D, T, B, 4, and 5.

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syn.2 The Language of Basic Modal Logic

Definição syn.1. The basic language of modal logic contains

1. The propositional constant for **falsidade** $\perp$.
2. The propositional constant for **verdade** $\top$.
3. A **infinitos enumeráveis** set of **variáveis proposicionais**: $p_0, p_1, p_2, \dots$

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4. The propositional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (condicional), $\leftrightarrow$ (bicondicional).
5. The modal operator $\Box$.
6. The modal operator $\Diamond$.

Definição syn.2. *Fórmulas* of the basic modal language are inductively defined as follows:

1. $\perp$ is an atomic fórmula.
2. $\top$ is an atomic fórmula.
3. Every variável proposicional p_i is an (atomic) fórmula.
4. If φ is a fórmula, then $\neg\varphi$ is a fórmula.
5. If φ and ψ are fórmulas, then $(\varphi \wedge \psi)$ is a fórmula.
6. If φ and ψ are fórmulas, then $(\varphi \vee \psi)$ is a fórmula.
7. If φ and ψ are fórmulas, then $(\varphi \rightarrow \psi)$ is a fórmula.
8. If φ and ψ are fórmulas, then $(\varphi \leftrightarrow \psi)$ is a fórmula.
9. If φ is a fórmula, then $\Box\varphi$ is a fórmula.
10. If φ is a fórmula, then $\Diamond\varphi$ is a fórmula.
11. Nothing else is a fórmula.

If a fórmula φ does not contain $\Box$ or $\Diamond$, we say it is *modal-free*.
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syn.3 Simultaneous Substitution

nml:syn:sub: sec An *instance* of a fórmula φ is the result of replacing all occurrences of a variável proposicional in φ by some other fórmula. We will refer to instances of fórmulas often, both when discussing validity and when discussing derivabilidade. It therefore is useful to define the notion precisely.

nml:syn:sub: def:subst-inst **Definição syn.3.** Where φ is a modal fórmula all of whose variáveis proposicionais are among $p_1, \dots, p_n$, and $\theta_1, \dots, \theta_n$ are also modal fórmulas, we define $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ as the result of simultaneously substituting each θ_i for p_i in φ . Formally, this is a definition by induction on φ :

1. $\varphi \equiv \perp$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\perp$.
2. $\varphi \equiv \top$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\top$.
3. $\varphi \equiv q$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is q , provided $q \neq p_i$ for $i = 1, \dots, n$.

4. $\varphi \equiv p_i$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is θ_i .
5. $\varphi \equiv \neg\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\neg\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
6. $\varphi \equiv (\psi \wedge \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
 $(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \wedge \chi[\theta_1/p_1, \dots, \theta_n/p_n])$.
7. $\varphi \equiv (\psi \vee \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
 $(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \vee \chi[\theta_1/p_1, \dots, \theta_n/p_n])$.
8. $\varphi \equiv (\psi \rightarrow \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
 $(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \rightarrow \chi[\theta_1/p_1, \dots, \theta_n/p_n])$.
9. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is
 $(\psi[\theta_1/p_1, \dots, \theta_n/p_n] \leftrightarrow \chi[\theta_1/p_1, \dots, \theta_n/p_n])$.
10. $\varphi \equiv \Box\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Box\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.
11. $\varphi \equiv \Diamond\psi$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\Diamond\psi[\theta_1/p_1, \dots, \theta_n/p_n]$.

The **fórmula** $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is called a *substitution instance* of φ .

Exemplo syn.4. Suppose φ is $p_1 \rightarrow \Box(p_1 \wedge p_2)$, θ_1 is $\Diamond(p_2 \rightarrow p_3)$ and θ_2 is $\neg\Box p_1$. Then $\varphi[\theta_1/p_1, \theta_2/p_2]$ is

$$\Diamond(p_2 \rightarrow p_3) \rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \wedge \neg\Box p_1)$$

while $\varphi[\theta_2/p_1, \theta_1/p_2]$ is

$$\neg\Box p_1 \rightarrow \Box(\neg\Box p_1 \wedge \Diamond(p_2 \rightarrow p_3))$$

Note that simultaneous substitution is in general not the same as iterated substitution, p.ex., compare $\varphi[\theta_1/p_1, \theta_2/p_2]$ above with $(\varphi[\theta_1/p_1])[\theta_2/p_2]$, which is:

$$\begin{aligned} \Diamond(p_2 \rightarrow p_3) &\rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \wedge p_2)[\neg\Box p_1/p_2], \text{ O.s.,} \\ \Diamond(\neg\Box p_1 \rightarrow p_3) &\rightarrow \Box(\Diamond(\neg\Box p_1 \rightarrow p_3) \wedge \neg\Box p_1) \end{aligned}$$

and with $(\varphi[\theta_2/p_2])[\theta_1/p_1]$:

$$\begin{aligned} p_1 &\rightarrow \Box(p_1 \wedge \neg\Box p_1)[\Diamond(p_2 \rightarrow p_3)/p_1], \text{ O.s.,} \\ \Diamond(p_2 \rightarrow p_3) &\rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \wedge \neg\Box\Diamond(p_2 \rightarrow p_3)). \end{aligned}$$

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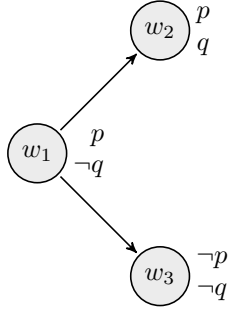


Figura syn.1: A simple model.

nml:syn:rel:
fig:simple

syn.4 Relational Models

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sec

The basic concept of semantics for normal modal logics is that of a *relational model*. It consists of a set of worlds, which are related by a binary “accessibility relation,” together with an assignment which determines which **variáveis proposicionais** count as “true” at which worlds.

Definição syn.5. A *model* for the basic modal language is a triple $\mathfrak{M} = \langle W, R, V \rangle$, where

1. W is a nonempty set of “worlds,”
2. R is a binary accessibility relation on W , and
3. V is a function assigning to each **variável proposicional** p a set $V(p)$ of possible worlds.

When Rww' holds, we say that w' is *accessible from* w . When $w \in V(p)$ we say p is *true at* w .

The great advantage of relational semantics is that models can be represented by means of simple diagrams, such as the one in **Figura syn.1**. Worlds are represented by nodes, and world w' is accessible from w precisely when there is an arrow from w to w' . Moreover, we label a node (world) by p when $w \in V(p)$, and otherwise by $\neg p$. **Figura syn.1** represents the model with $W = \{w_1, w_2, w_3\}$, $R = \{\langle w_1, w_2 \rangle, \langle w_1, w_3 \rangle\}$, $V(p) = \{w_1, w_2\}$, and $V(q) = \{w_2\}$.

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syn.5 Truth at a World

nml:syn:trw:
sec

Every modal model determines which modal **fórmulas** count as true at which worlds in it. The relation “model $\mathfrak{M}$ makes **fórmula** φ true at world w ” is the

basic notion of relational semantics. The relation is defined inductively and coincides with the usual characterization using truth tables for the non-modal operators.

Definição syn.6. *Truth of a fórmula φ at w in a $\mathfrak{M}$, in symbols: $\mathfrak{M}, w \Vdash \varphi$, is defined inductively as follows:*

[nml:syn:trw:](#)
[defn:mmodels](#)

1. $\varphi \equiv \perp$: Never $\mathfrak{M}, w \Vdash \perp$.
2. $\varphi \equiv \top$: Always $\mathfrak{M}, w \Vdash \top$.
3. $\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$.
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \not\Vdash \psi$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$.
6. $\varphi \equiv (\psi \vee \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \not\Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff either both $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$ or neither $\mathfrak{M}, w \Vdash \psi$ nor $\mathfrak{M}, w \Vdash \chi$.
9. $\varphi \equiv \Box\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w' \Vdash \psi$ for all $w' \in W$ with Rww' .
10. $\varphi \equiv \Diamond\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w' \Vdash \psi$ for at least one $w' \in W$ with Rww' .

[nml:syn:trw:](#)
[defn:sub:mmodels-box](#)

[nml:syn:trw:](#)
[defn:sub:mmodels-diamond](#)

Note that by clause (9), a fórmula $\Box\psi$ is true at w whenever there are no w' with Rww' . In such a case $\Box\psi$ is *vacuously* true at w . Also, $\Box\psi$ may be satisfied at w even if ψ is not. The truth of ψ at w does not guarantee the truth of $\Diamond\psi$ at w . This holds, however, if Rww , p.ex., if R is reflexive. If there is no w' such that Rww' , then $\mathfrak{M}, w \not\Vdash \Diamond\varphi$, for any φ .

Problema syn.1. Consider the model of [Figura syn.1](#). Which of the following hold?

1. $\mathfrak{M}, w_1 \Vdash q$;
2. $\mathfrak{M}, w_3 \Vdash \neg q$;
3. $\mathfrak{M}, w_1 \Vdash p \vee q$;
4. $\mathfrak{M}, w_1 \Vdash \Box(p \vee q)$;
5. $\mathfrak{M}, w_3 \Vdash \Box q$;
6. $\mathfrak{M}, w_3 \Vdash \Box \perp$;
7. $\mathfrak{M}, w_1 \Vdash \Diamond q$;
8. $\mathfrak{M}, w_1 \Vdash \Box q$;

9. $\mathfrak{M}, w_1 \Vdash \neg \Box \Box \neg q$.

nml:syn:tru: **Proposição syn.7.**
prop:dual

1. $\mathfrak{M}, w \Vdash \Box \varphi$ iff $\mathfrak{M}, w \Vdash \neg \Diamond \neg \varphi$.
2. $\mathfrak{M}, w \Vdash \Diamond \varphi$ iff $\mathfrak{M}, w \Vdash \neg \Box \neg \varphi$.

Prova. 1. $\mathfrak{M}, w \Vdash \neg \Diamond \neg \varphi$ iff $\mathfrak{M}, w \not\Vdash \Diamond \neg \varphi$ by definition of $\mathfrak{M}, w \Vdash$. $\mathfrak{M}, w \Vdash \Diamond \neg \varphi$ iff for some w' with Rww' , $\mathfrak{M}, w' \Vdash \neg \varphi$. Hence, $\mathfrak{M}, w \not\Vdash \Diamond \neg \varphi$ iff for all w' with Rww' , $\mathfrak{M}, w' \not\Vdash \neg \varphi$. We also have $\mathfrak{M}, w' \not\Vdash \neg \varphi$ iff $\mathfrak{M}, w' \Vdash \varphi$. Together we have $\mathfrak{M}, w \Vdash \neg \Diamond \neg \varphi$ iff for all w' with Rww' , $\mathfrak{M}, w' \Vdash \varphi$. Again by definition of $\mathfrak{M}, w \Vdash$, that is the case iff $\mathfrak{M}, w \Vdash \Box \varphi$.

2. $\mathfrak{M}, w \Vdash \neg \Box \neg \varphi$ iff $\mathfrak{M} \not\Vdash \Box \neg \varphi$. $\mathfrak{M}, w \Vdash \Box \neg \varphi$ iff for all w' with Rww' , $\mathfrak{M}, w' \Vdash \neg \varphi$. Hence, $\mathfrak{M}, w \not\Vdash \Box \neg \varphi$ iff for some w' with Rww' , $\mathfrak{M}, w' \not\Vdash \neg \varphi$. We also have $\mathfrak{M}, w' \not\Vdash \neg \varphi$ iff $\mathfrak{M}, w' \Vdash \varphi$. Together we have $\mathfrak{M}, w \Vdash \neg \Box \neg \varphi$ iff for some w' with Rww' , $\mathfrak{M}, w' \Vdash \varphi$. Again by definition of $\mathfrak{M}, w \Vdash$, that is the case iff $\mathfrak{M}, w \Vdash \Diamond \varphi$. $\square$

Problema syn.2. Complete the proof of **Proposição syn.7**.

Problema syn.3. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model, and suppose $w_1, w_2 \in W$ are such that:

1. $w_1 \in V(p)$ if and only if $w_2 \in V(p)$ (for every **variável proposicional** p);
and
2. for all $w \in W$: Rw_1w if and only if Rw_2w .

Using induction on **fórmulas**, show that for all **fórmulas** φ : $\mathfrak{M}, w_1 \Vdash \varphi$ if and only if $\mathfrak{M}, w_2 \Vdash \varphi$.

Problema syn.4. Let $\mathfrak{M} = \langle W, R, V \rangle$. Show that $\mathfrak{M}, w \Vdash \neg \Diamond \varphi$ if and only if $\mathfrak{M}, w \Vdash \Box \neg \varphi$.

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syn.6 Truth in a Model

nml:syn:tru: Sometimes we are interested in which **fórmulas** are true at every world in a
sec given model. Let's introduce a notation for this.

Definição syn.8. A **fórmula** φ is *true in a model* $M = \langle W, R, V \rangle$, written $\mathfrak{M} \Vdash \varphi$, if and only if $\mathfrak{M}, w \Vdash \varphi$ for every $w \in W$.

nml:syn:tru: **Proposição syn.9.**
prop:truthfacts

1. If $\mathfrak{M} \Vdash \varphi$ then $\mathfrak{M} \not\Vdash \neg \varphi$, but not vice-versa.

2. If $\mathfrak{M} \Vdash \varphi \rightarrow \psi$ then $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but not vice-versa.

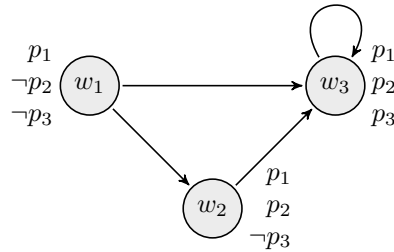
Prova. 1. If $\mathfrak{M} \Vdash \varphi$ then φ is true at all worlds in W , and since $W \neq \emptyset$, it can't be that $\mathfrak{M} \Vdash \neg\varphi$, or else φ would have to be both true and false at some world.

On the other hand, if $\mathfrak{M} \not\Vdash \neg\varphi$ then φ is true at some world $w \in W$. It does not follow that $\mathfrak{M}, w \Vdash \varphi$ for *every* $w \in W$. For instance, in the model of [Figura syn.1](#), $\mathfrak{M} \not\Vdash p$, and also $\mathfrak{M} \not\Vdash p$.

2. Assume $\mathfrak{M} \Vdash \varphi \rightarrow \psi$ and $\mathfrak{M} \Vdash \varphi$; to show $\mathfrak{M} \Vdash \psi$ let $w \in W$ be an arbitrary world. Then $\mathfrak{M}, w \Vdash \varphi \rightarrow \psi$ and $\mathfrak{M}, w \Vdash \varphi$, so $\mathfrak{M}, w \Vdash \psi$, and since w was arbitrary, $\mathfrak{M} \Vdash \psi$.

To show that the converse fails, we need to find a model $\mathfrak{M}$ such that $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but $\mathfrak{M} \not\Vdash \varphi \rightarrow \psi$. Consider again the model of [Figura syn.1](#): $\mathfrak{M} \not\Vdash p$ and hence (vacuously) $\mathfrak{M} \Vdash p$ only if $\mathfrak{M} \Vdash q$. However, $\mathfrak{M} \not\Vdash p \rightarrow q$, as p is true but q false at w_1 . $\square$

Problema syn.5. Consider the following model $\mathfrak{M}$ for the language comprising p_1, p_2, p_3 as the only **variáveis proposicionais**:



Are the following **fórmulas** and schemas true in the model $\mathfrak{M}$, O.s., true at every world in $\mathfrak{M}$? Explain.

1. $p \rightarrow \Diamond p$ (for p atomic);
2. $\varphi \rightarrow \Diamond \varphi$ (for φ arbitrary);
3. $\Box p \rightarrow p$ (for p atomic);
4. $\neg p \rightarrow \Diamond \Box p$ (for p atomic);
5. $\Diamond \Box \varphi$ (for φ arbitrary);
6. $\Box \Diamond p$ (for p atomic).

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syn.7 Validity

nml:syn:val:
sec

Fórmulas that are true in all models, O.s., true at every world in every model, are particularly interesting. They represent those modal propositions which are true regardless of how $\Box$ and $\Diamond$ are interpreted, as long as the interpretation is “normal” in the sense that it is generated by some accessibility relation on possible worlds. We call such **fórmulas** *valid*. For instance, $\Box(p \wedge q) \rightarrow \Box p$ is valid. Some **fórmulas** one might expect to be valid on the basis of the alethic interpretation of $\Box$, such as $\Box p \rightarrow p$, are not valid, however. Part of the interest of relational models is that different interpretations of $\Box$ and $\Diamond$ can be captured by different kinds of accessibility relations. This suggests that we should define validity not just relative to *all* models, but relative to all models of *a certain kind*. It will turn out, p.ex., that $\Box p \rightarrow p$ is true in all models where every world is accessible from itself, O.s., R is reflexive. Defining validity relative to classes of models enables us to formulate this succinctly: $\Box p \rightarrow p$ is valid in the class of reflexive models.

explanation

Definição syn.10. A **fórmula** φ is *valid* in a class $\mathcal{C}$ of models if it is true in every model in $\mathcal{C}$ (O.s., true at every world in every model in $\mathcal{C}$). If φ is valid in $\mathcal{C}$, we write $\mathcal{C} \models \varphi$, and we write $\models \varphi$ if φ is valid in the class of *all* models.

nml:syn:val:
prop:subset-class

Proposição syn.11. If φ is valid in $\mathcal{C}$ it is also valid in each class $\mathcal{C}' \subseteq \mathcal{C}$.

nml:syn:val:
prop:Nec-rule

Proposição syn.12. If φ is valid, then so is $\Box\varphi$.

Prova. Assume $\models \varphi$. To show $\models \Box\varphi$ let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and $w \in W$. If Rww' then $\mathfrak{M}, w' \models \varphi$, since φ is valid, and so also $\mathfrak{M}, w \models \Box\varphi$. Since $\mathfrak{M}$ and w were arbitrary, $\models \Box\varphi$. $\square$

Problema syn.6. Show that the following are valid:

1. $\Box p \rightarrow \Box(q \rightarrow p)$;
2. $\Box \neg \perp$;
3. $\Box p \rightarrow (\Box q \rightarrow \Box p)$.

Problema syn.7. Show that $\varphi \rightarrow \Box\varphi$ is valid in the class $\mathcal{C}$ of models $\mathfrak{M} = \langle W, R, V \rangle$ where $W = \{w\}$. Similarly, show that $\psi \rightarrow \Box\varphi$ and $\Diamond\varphi \rightarrow \psi$ are valid in the class of models $\mathfrak{M} = \langle W, R, V \rangle$ where $R = \emptyset$.

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syn.8 Tautological Instances

explanation

A modal-free formula is a tautology if it is true under every truth-value assignment. Clearly, every tautology is true at every world in every model. But for **fórmulas** involving $\Box$ and $\Diamond$, the notion of tautology is not defined. Is it the case, p.ex., that $\Box p \vee \neg \Box p$ —an instance of the principle of excluded middle—is valid? The notion of a *tautological instance* helps: a **fórmula** that is a substitution instance of a (non-modal) tautology. It is not surprising, but still requires proof, that every tautological instance is valid.

nml:syn:tau:
sec

Definição syn.13. A modal **fórmula** ψ is a *tautological instance* if and only if there is a modal-free tautology φ with **variáveis proposicionais** $p_1, \dots, p_n$ and **fórmulas** $\theta_1, \dots, \theta_n$ such that $\psi \equiv \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.

Lema syn.14. Suppose φ is a modal-free **fórmula** whose **variáveis proposicionais** are $p_1, \dots, p_n$, and let $\theta_1, \dots, \theta_n$ be modal **fórmulas**. Then for any assignment $\mathbf{v}$, any model $\mathfrak{M} = \langle W, R, V \rangle$, and any $w \in W$ such that $\mathbf{v}(p_i) = \mathbb{T}$ if and only if $\mathfrak{M}, w \Vdash \theta_i$ we have that $\mathbf{v} \models \varphi$ if and only if $\mathfrak{M}, w \Vdash \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.

nml:syn:tau:
lem:valid-taut

Prova. By induction on φ .

1. $\varphi \equiv \perp$: Both $\mathbf{v} \not\models \perp$ and $\mathfrak{M}, w \not\Vdash \perp$.
2. $\varphi \equiv \top$: Both $\mathbf{v} \models \top$ and $\mathfrak{M}, w \Vdash \top$.
3. $\varphi \equiv p_i$:

$$\begin{aligned}
 \mathbf{v} \models p_i &\Leftrightarrow \mathbf{v}(p_i) = \mathbb{T} \\
 &\quad \text{by definition of } \mathbf{v} \models p_i \\
 &\Leftrightarrow \mathfrak{M}, w \Vdash \theta_i \\
 &\quad \text{by assumption} \\
 &\Leftrightarrow \mathfrak{M}, w \Vdash p_i[\theta_1/p_1, \dots, \theta_n/p_n] \\
 &\quad \text{since } p_i[\theta_1/p_1, \dots, \theta_n/p_n] \equiv \theta_i.
 \end{aligned}$$

4. $\varphi \equiv \neg\psi$:

$$\begin{aligned}
 \mathbf{v} \models \neg\psi &\Leftrightarrow \mathbf{v} \not\models \psi \\
 &\quad \text{by definition of } \mathbf{v} \models; \\
 &\Leftrightarrow \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \\
 &\quad \text{by induction hypothesis} \\
 &\Leftrightarrow \mathfrak{M}, w \Vdash \neg\psi[\theta_1/p_1, \dots, \theta_n/p_n] \\
 &\quad \text{by definition of } \mathbf{v} \models.
 \end{aligned}$$

5. $\varphi \equiv (\psi \wedge \chi)$:

$$\begin{aligned}
\mathfrak{v} \models \psi \wedge \chi &\Leftrightarrow \mathfrak{v} \models \psi \text{ and } \mathfrak{v} \models \chi \\
&\text{by definition of } \mathfrak{v} \models \\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \wedge \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

6. $\varphi \equiv (\psi \vee \chi)$:

$$\begin{aligned}
\mathfrak{v} \models \psi \vee \chi &\Leftrightarrow \mathfrak{v} \models \psi \text{ or } \mathfrak{v} \models \chi \\
&\text{by definition of } \mathfrak{v} \models; \\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \vee \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

7. $\varphi \equiv (\psi \rightarrow \chi)$:

$$\begin{aligned}
\mathfrak{v} \models \psi \rightarrow \chi &\Leftrightarrow \mathfrak{v} \not\models \psi \text{ or } \mathfrak{v} \models \chi \\
&\text{by definition of } \mathfrak{v} \models \\
&\Leftrightarrow \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \rightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

8. $\varphi \equiv (\psi \leftrightarrow \chi)$:

$$\begin{aligned}
\mathfrak{v} \models \psi \leftrightarrow \chi &\Leftrightarrow \text{either } \mathfrak{v} \models \psi \text{ and } \mathfrak{v} \models \chi \\
&\quad \text{or } \mathfrak{v} \not\models \psi \text{ and } \mathfrak{v} \not\models \chi \\
&\text{by definition of } \mathfrak{v} \models \\
&\Leftrightarrow \text{either } \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\quad \text{or } \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \not\Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \leftrightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

□

Proposição syn.15. *All tautological instances are valid.*

*nml:syn:tau:
prop:valid-taut*

Prova. Contrapositively, suppose φ is such that $\mathfrak{M}, w \not\models \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$, for some model $\mathfrak{M}$ and world w . Define an assignment $\mathfrak{v}$ such that $\mathfrak{v}(p_i) = \mathbb{T}$ if and only if $\mathfrak{M}, w \models \theta_i$ (and $\mathfrak{v}$ assigns arbitrary values to $q \notin \{p_1, \dots, p_n\}$). Then by **Lema syn.14**, $\mathfrak{v} \not\models \varphi$, so φ is not a tautology. $\square$

-NoValue-

syn.9 Schemas and Validity

Definição syn.16. A *schema* is a set of **fórmulas** comprising all and only the substitution instances of some modal **fórmula** χ , O.s.,

*nml:syn:sch:
sec*

$$\{\psi : \exists \theta_1, \dots, \exists \theta_n (\psi = \chi[\theta_1/p_1, \dots, \theta_n/p_n])\}.$$

The **fórmula** χ is called the *characteristic fórmula* of the schema, and it is unique up to a renaming of the **variáveis proposicionais**. A **fórmula** φ is an *instance* of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting ‘ φ ’, ‘ ψ ’, $\dots$, for the atomic components of χ . So, for instance, the following denote schemas: ‘ φ ’, ‘ $\varphi \rightarrow \Box \varphi$ ’, ‘ $\varphi \rightarrow (\psi \rightarrow \varphi)$ ’. They correspond to the characteristic **fórmulas** p , $p \rightarrow \Box p$, $p \rightarrow (q \rightarrow p)$. The schema ‘ φ ’ denotes the set of *all fórmulas*.

Definição syn.17. A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

Proposição syn.18. *The following schema K is valid*

*nml:syn:sch:
prop:Kvalid*

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi). \quad (\text{K})$$

Prova. We need to show that all instances of the schema are true at every world in every model. So let $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that the consequent is true as well. In this case, let $\mathfrak{M}, w \models \Box(\varphi \rightarrow \psi)$ and $\mathfrak{M}, w \models \Box \varphi$. We need to show $\mathfrak{M}, w \models \Box \psi$. So let w' be arbitrary such that Rww' . Then by the first assumption $\mathfrak{M}, w' \models \varphi \rightarrow \psi$ and by the second assumption $\mathfrak{M}, w' \models \varphi$. It follows that $\mathfrak{M}, w' \models \psi$. Since w' was arbitrary, $\mathfrak{M}, w \models \Box \psi$. $\square$

Proposição syn.19. *The following schema DUAL is valid*

*nml:syn:sch:
prop:Dual-valid*

$$\Diamond \varphi \leftrightarrow \neg \Box \neg \varphi. \quad (\text{DUAL})$$

Prova. Exercise. $\square$

Problema syn.8. Prove **Proposição syn.19**.

*nml:syn:sch:
prop:soundMP*

Proposição syn.20. If φ and $\varphi \rightarrow \psi$ are true at a world in a model then so is ψ . Hence, the valid *fórmulas* are closed under modus ponens.

*nml:syn:sch:
prop:valid-instances*

Proposição syn.21. A *fórmula* φ is valid iff all its substitution instances are. In other words, a schema is valid iff its characteristic *fórmula* is.

Prova. The “if” direction is obvious, since φ is a substitution instance of itself.

To prove the “only if” direction, we show the following: Suppose $\mathfrak{M} = \langle W, R, V \rangle$ is a modal model, and $\psi \equiv \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is a substitution instance of φ . Define $\mathfrak{M}' = \langle W, R, V' \rangle$ by $V'(p_i) = \{w : \mathfrak{M}, w \Vdash \theta_i\}$. Then $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}', w \Vdash \varphi$, for any $w \in W$. (We leave the proof as an exercise.) Now suppose that φ was valid, but some substitution instance ψ of φ was not valid. Then for some $\mathfrak{M} = \langle W, R, V \rangle$ and some $w \in W$, $\mathfrak{M}, w \not\Vdash \psi$. But then $\mathfrak{M}', w \not\Vdash \varphi$ by the claim, and φ is not valid, a contradiction. $\square$

Problema syn.9. Prove the claim in the “only if” part of the proof of **Proposição syn.21**. (Hint: use induction on φ .)

Note, however, that it is not true that a schema is true in a model iff its characteristic formula is. Of course, the “only if” direction holds: if every instance of φ is true in $\mathfrak{M}$, φ itself is true in $\mathfrak{M}$. But it may happen that φ is true in $\mathfrak{M}$ but some instance of φ is false at some world in $\mathfrak{M}$. For a very simple counterexample consider p in a model with only one world w and $V(p) = \{w\}$, so that p is true at w . But $\perp$ is an instance of p , and not true at w .

Problema syn.10. Show that none of the following *fórmulas* are valid:

- D: $\Box p \rightarrow \Diamond p$;
- T: $\Box p \rightarrow p$;
- B: $p \rightarrow \Box \Diamond p$;
- 4: $\Box p \rightarrow \Box \Box p$;
- 5: $\Diamond p \rightarrow \Box \Diamond p$.

Problema syn.11. Prove that the schemas in the first column of **Tabela syn.1** are valid and those in the second column are not valid.

Problema syn.12. Decide whether the following schemas are valid or invalid:

1. $(\Diamond \varphi \rightarrow \Box \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$;
2. $\Diamond(\varphi \rightarrow \psi) \vee \Box(\psi \rightarrow \varphi)$.

Problema syn.13. For each of the following schemas find a model $\mathfrak{M}$ such that every instance of the *fórmula* is true in $\mathfrak{M}$:

Valid Schemas	Invalid Schemas
$\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$	$\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Box\psi)$
$\Diamond(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Diamond\psi)$	$(\Diamond\varphi \wedge \Diamond\psi) \rightarrow \Diamond(\varphi \wedge \psi)$
$\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$	$\varphi \rightarrow \Box\varphi$
$\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$	$\Box\Diamond\varphi \rightarrow \psi$
$\neg\Diamond\varphi \rightarrow \Box(\varphi \rightarrow \psi)$	$\Box\Box\varphi \rightarrow \Box\varphi$
$\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$	$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi.$

Tabela syn.1: Valid and (or?) invalid schemas.

nml:syn:sch:
tab:valid-invalidSchemas

1. $p \rightarrow \Diamond\Diamond p$;
2. $\Diamond p \rightarrow \Box p$.

-NoValue-

syn.10 Entailment

explanation With the definition of truth at a world, we can define an entailment relation between **fórmulas**. A **fórmula** ψ entails φ iff, whenever ψ is true, φ is true as well. Here, “whenever” means both “whichever model we consider” as well as “whichever world in that model we consider.”

nml:syn:ent:
sec

Definição syn.22. If Γ is a set of **fórmulas** and φ a **fórmula**, then Γ entails φ , in symbols: $\Gamma \models \varphi$, if and only if for every model $\mathfrak{M} = \langle W, R, V \rangle$ and world $w \in W$, if $\mathfrak{M}, w \models \psi$ for every $\psi \in \Gamma$, then $\mathfrak{M}, w \models \varphi$. If Γ contains a single **fórmula** ψ , then we write $\psi \models \varphi$.

Exemplo syn.23. To show that a **fórmula** entails another, we have to reason about all models, using the definition of $\mathfrak{M}, w \models$. For instance, to show $p \rightarrow \Diamond p \models \Box \neg p \rightarrow \neg p$, we might argue as follows: Consider a model $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$, and suppose $\mathfrak{M}, w \models p \rightarrow \Diamond p$. We have to show that $\mathfrak{M}, w \models \Box \neg p \rightarrow \neg p$. Suppose not. Then $\mathfrak{M}, w \models \Box \neg p$ and $\mathfrak{M}, w \not\models \neg p$. Since $\mathfrak{M}, w \not\models \neg p$, $\mathfrak{M}, w \models p$. By assumption, $\mathfrak{M}, w \models p \rightarrow \Diamond p$, hence $\mathfrak{M}, w \models \Diamond p$. By definition of $\mathfrak{M}, w \models \Diamond p$, there is some w' with Rww' such that $\mathfrak{M}, w' \models p$. Since also $\mathfrak{M}, w \models \Box \neg p$, $\mathfrak{M}, w' \models \neg p$, a contradiction.

To show that a **fórmula** ψ does not entail another φ , we have to give a counterexample, O.s., a model $\mathfrak{M} = \langle W, R, V \rangle$ where we show that at some world $w \in W$, $\mathfrak{M}, w \models \psi$ but $\mathfrak{M}, w \not\models \varphi$. Let's show that $p \rightarrow \Diamond p \not\models \Box p \rightarrow p$. Consider the model in **Figura syn.2**. We have $\mathfrak{M}, w_1 \models \Diamond p$ and hence $\mathfrak{M}, w_1 \models p \rightarrow \Diamond p$. However, since $\mathfrak{M}, w_1 \models \Box p$ but $\mathfrak{M}, w_1 \not\models p$, we have $\mathfrak{M}, w_1 \not\models \Box p \rightarrow p$.

Often very simple counterexamples suffice. The model $\mathfrak{M}' = \{W', R', V'\}$ with $W' = \{w\}$, $R' = \emptyset$, and $V'(p) = \emptyset$ is also a counterexample: Since $\mathfrak{M}', w \not\models p$, $\mathfrak{M}', w \models p \rightarrow \Diamond p$. As no worlds are accessible from w , we have $\mathfrak{M}', w \models \Box p$, and so $\mathfrak{M}', w \not\models \Box p \rightarrow p$.

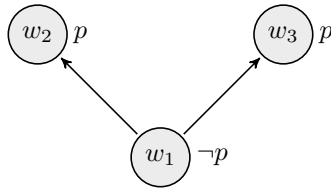


Figura syn.2: Counterexample to $p \rightarrow \Diamond p \models \Box p \rightarrow p$.

nml:syn:ent:
fig:counterex

Problema syn.14. Show that $\Box(\varphi \wedge \psi) \models \Box\varphi$.

Problema syn.15. Show that $\Box(p \rightarrow q) \not\models p \rightarrow \Box q$ and $p \rightarrow \Box q \not\models \Box(p \rightarrow q)$.

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Referências Bibliográficas