Parte I

Normal Modal Logics

This part covers the metatheory of normal modal logics. It currently consists of Aldo Antonelli's notes on classical correspondence theory for basic modal logic.

Capítulo 1

Syntax and Semantics

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1.1 Introduction

nml:syn:int: sec Modal logic deals with *modal propositions* and the entailment relations among them. Examples of modal propositions are the following:

1. It is necessary that $2 + 2 = 4$.
2. It is necessarily possible that it will rain tomorrow.
3. If it is necessarily possible that φ then it is possible that φ.

Possibility and necessity are not the only modalities: other unary connectives are also classified as modalities, for instance, "it ought to be the case that φ," "It will be the case that φ," "Dana knows that φ," or "Dana believes that φ."

Modal logic makes its first appearance in Aristotle's *De Interpretatione*: he was the first to notice that necessity implies possibility, but not vice versa; that possibility and necessity are inter-definable; that If $\varphi \land \psi$ is possibly true then φ is possibly true and ψ is possibly true, but not conversely; and that if $\varphi \to \psi$ is necessary, then if φ is necessary, so is ψ.

The first modern approach to modal logic was the work of C. I. Lewis, culminating with Lewis and Langford, *Symbolic Logic* (1932). Lewis & Langford were unhappy with the representation of implication by means of the material conditional: $\varphi \to \psi$ is a poor substitute for "φ implies ψ." Instead, they proposed to characterize implication as "Necessarily, if φ then ψ," symbolized as $\varphi \strictif \psi$. In trying to sort out the different properties, Lewis identified five different modal systems, **S1**, ..., **S4**, **S5**, the last two of which are still in use.

The approach of Lewis and Langford was purely *syntactical*: they identified reasonable axioms and rules and investigated what was provable with those means. A semantic approach remained elusive for a long time, until a first attempt was made by Rudolf Carnap in *Meaning and Necessity* (1947) using the notion of a *state description*, O.s., a collection of atomic sentences (those that are “true” in that state description). After lifting the truth definition to arbitrary sentences φ, Carnap defines φ to be *necessarily true* if it is true in all state descriptions. Carnap's approach could not handle *iterated* modalities, in that sentences of the form “Possibly necessarily ... possibly φ” always reduce to the innermost modality.

The major breakthrough in modal semantics came with Saul Kripke's article “A Completeness Theorem in Modal Logic” (JSL 1959). Kripke based his work on Leibniz's idea that a statement is necessarily true if it is true “at all possible worlds.” This idea, though, suffers from the same drawbacks as Carnap's, in that the truth of statement at a world w (or a state description s) does not depend on w at all. So Kripke assumed that worlds are related by an *accessibility relation* R, and that a statement of the form “Necessarily φ” is true at a world w if and only if φ is true at all worlds w' *accessible from* w. Semantics that provide some version of this approach are called Kripke semantics and made possible the tumultuous development of modal logics (in the plural).

When interpreted by the Kripke semantics, modal logic shows us what *relational structures* look like “from the inside.” A relational structure is just a set equipped with a binary relation (for instance, the set of students in the class ordered by their social security number is a relational structure). But in fact relational structures come in all sorts of domains: besides relative possibility of states of the world, we can have epistemic states of some agent related by epistemic possibility, or states of a dynamical system with their state transitions, etc. Modal logic can be used to model all of these: the first gives us ordinary, alethic, modal logic; the others give us epistemic logic, dynamic logic, etc.

We focus on one particular angle, known to modal logicians as “correspondence theory.” One of the most significant early discoveries of Kripke's is that many properties of the accessibility relation R (whether it is transitive, symmetric, etc.) can be characterized *in the modal language* itself by means of appropriate “modal schemas.” Modal logicians say, for instance, that the reflexivity of R “corresponds” to the schema “If necessarily φ, then φ”. We explore mainly the correspondence theory of a number of classical systems of modal logic (p.ex., **S4** and **S5**) obtained by a combination of the schemas D, T, B, 4, and 5.

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1.2 The Language of Basic Modal Logic

Definição 1.1. The basic language of modal logic contains nml:syn:lan:sec

1. The propositional constant for falsidade $\bot$.
2. The propositional constant for verdade $\top$.
3. A infinitos enumeráveis set of variáveis proposicionais: p_0, p_1, p_2, ...

4. The propositional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (condicional), $\leftrightarrow$ (bicondicional).

5. The modal operator $\Box$.

6. The modal operator $\Diamond$.

Definição 1.2. *Fórmulas* of the basic modal language are inductively defined as follows:

1. $\bot$ is an atomic fórmula.
2. $\top$ is an atomic fórmula.
3. Every variável proposicional p_i is an (atomic) fórmula.
4. If φ is a fórmula, then $\neg\varphi$ is a fórmula.
5. If φ and ψ are fórmulas, then $(\varphi \wedge \psi)$ is a fórmula.
6. If φ and ψ are fórmulas, then $(\varphi \vee \psi)$ is a fórmula.
7. If φ and ψ are fórmulas, then $(\varphi \rightarrow \psi)$ is a fórmula.
8. If φ and ψ are fórmulas, then $(\varphi \leftrightarrow \psi)$ is a fórmula.
9. If φ is a fórmula, then $\Box\varphi$ is a fórmula.
10. If φ is a fórmula, then $\Diamond\varphi$ is a fórmula.
11. Nothing else is a fórmula.

If a fórmula φ does not contain $\Box$ or $\Diamond$, we say it is *modal-free*.
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1.3 Simultaneous Substitution

nml:syn:sub:sec An *instance* of a fórmula φ is the result of replacing all occurrences of a variável proposicional in φ by some other fórmula. We will refer to instances of fórmulas often, both when discussing validity and when discussing derivabilidade. It therefore is useful to define the notion precisely.

nml:syn:sub:def:subst-inst **Definição 1.3.** Where φ is a modal fórmula all of whose variáveis proposicionais are among $p_1, \dots, p_n$, and $\theta_1, \dots, \theta_n$ are also modal fórmulas, we define $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ as the result of simultaneously substituting each θ_i for p_i in φ. Formally, this is a definition by induction on φ:

1. $\varphi \equiv \bot$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\bot$.
2. $\varphi \equiv \top$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is $\top$.
3. $\varphi \equiv q$: $\varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is q, provided $q \not\equiv p_i$ for $i = 1, \dots, n$.

4. $\varphi \equiv p_i$: $\varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$ is θ_i.
5. $\varphi \equiv \neg\psi$: $\varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$ is $\neg\psi[\theta_1/p_1, \ldots, \theta_n/p_n]$.
6. $\varphi \equiv (\psi \land \chi)$: $\varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \ldots, \theta_n/p_n] \land \chi[\theta_1/p_1, \ldots, \theta_n/p_n]).$$
7. $\varphi \equiv (\psi \lor \chi)$: $\varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \ldots, \theta_n/p_n] \lor \chi[\theta_1/p_1, \ldots, \theta_n/p_n]).$$
8. $\varphi \equiv (\psi \rightarrow \chi)$: $\varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \ldots, \theta_n/p_n] \rightarrow \chi[\theta_1/p_1, \ldots, \theta_n/p_n]).$$
9. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$ is
$$(\psi[\theta_1/p_1, \ldots, \theta_n/p_n] \leftrightarrow \chi[\theta_1/p_1, \ldots, \theta_n/p_n]).$$
10. $\varphi \equiv \Box\psi$: $\varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$ is $\Box\psi[\theta_1/p_1, \ldots, \theta_n/p_n]$.
11. $\varphi \equiv \Diamond\psi$: $\varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$ is $\Diamond\psi[\theta_1/p_1, \ldots, \theta_n/p_n]$.

The fórmula $\varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$ is called a *substitution instance* of φ.

Exemplo 1.4. Suppose φ is $p_1 \rightarrow \Box(p_1 \land p_2)$, θ_1 is $\Diamond(p_2 \rightarrow p_3)$ and θ_2 is $\neg\Box p_1$. Then $\varphi[\theta_1/p_1, \theta_2/p_2]$ is

$$\Diamond(p_2 \rightarrow p_3) \rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \land \neg\Box p_1)$$

while $\varphi[\theta_2/p_1, \theta_1/p_2]$ is

$$\neg\Box p_1 \rightarrow \Box(\neg\Box p_1 \land \Diamond(p_2 \rightarrow p_3))$$

Note that simultaneous substitution is in general not the same as iterated substitution, p.ex., compare $\varphi[\theta_1/p_1, \theta_2/p_2]$ above with $(\varphi[\theta_1/p_1])[\theta_2/p_2]$, which is:

$$\begin{aligned} \Diamond(p_2 \rightarrow p_3) \rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \land p_2)[\neg\Box p_1/p_2], \text{ O.s.,} \\ \Diamond(\neg\Box p_1 \rightarrow p_3) \rightarrow \Box(\Diamond(\neg\Box p_1 \rightarrow p_3) \land \neg\Box p_1) \end{aligned}$$

and with $(\varphi[\theta_2/p_2])[\theta_1/p_1]$:

$$\begin{aligned} p_1 \rightarrow \Box(p_1 \land \neg\Box p_1)[\Diamond(p_2 \rightarrow p_3)/p_1], \text{ O.s.,} \\ \Diamond(p_2 \rightarrow p_3) \rightarrow \Box(\Diamond(p_2 \rightarrow p_3) \land \neg\Box\Diamond(p_2 \rightarrow p_3)). \end{aligned}$$

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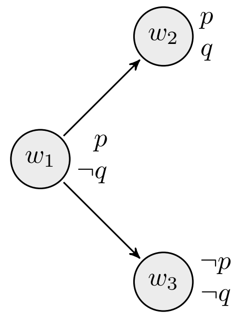

Figura 1.1: A simple model.

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fig:simple

1.4 Relational Models

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sec

The basic concept of semantics for normal modal logics is that of a *relational model*. It consists of a set of worlds, which are related by a binary "accessibility relation," together with an assignment which determines which variáveis proposicionais count as "true" at which worlds.

Definição 1.5. A *model* for the basic modal language is a triple $\mathfrak{M} = \langle W, R, V \rangle$, where

1. W is a nonempty set of "worlds,"
2. R is a binary accessibility relation on W, and
3. V is a function assigning to each variável proposicional p a set $V(p)$ of possible worlds.

When Rww' holds, we say that w' is *accessible from* w. When $w \in V(p)$ we say p is *true at* w.

The great advantage of relational semantics is that models can be represented by means of simple diagrams, such as the one in Figura 1.1. Worlds are represented by nodes, and world w' is accessible from w precisely when there is an arrow from w to w'. Moreover, we label a node (world) by p when $w \in V(p)$, and otherwise by $\neg p$. Figura 1.1 represents the model with $W = \{w_1, w_2, w_3\}$, $R = \{\langle w_1, w_2\rangle, \langle w_1, w_3\rangle\}$, $V(p) = \{w_1, w_2\}$, and $V(q) = \{w_2\}$.

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1.5 Truth at a World

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sec

Every modal model determines which modal fórmulas count as true at which worlds in it. The relation "model $\mathfrak{M}$ makes fórmula φ true at world w" is the basic notion of relational semantics. The relation is defined inductively and coincides with the usual characterization using truth tables for the non-modal operators.

Definição 1.6. *Truth of a fórmula* φ *at* w in a $\mathfrak{M}$, in symbols: $\mathfrak{M}, w \Vdash \varphi$, is defined inductively as follows: nml:syn:trw: defn:mmodels

1. $\varphi \equiv \bot$: Never $\mathfrak{M}, w \Vdash \bot$.
2. $\varphi \equiv \top$: Always $\mathfrak{M}, w \Vdash \top$.
3. $\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$.
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \nVdash \psi$.
5. $\varphi \equiv (\psi \land \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$.
6. $\varphi \equiv (\psi \lor \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$ (or both).
7. $\varphi \equiv (\psi \to \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \nVdash \psi$ or $\mathfrak{M}, w \Vdash \chi$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff either both $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$ or neither $\mathfrak{M}, w \Vdash \psi$ nor $\mathfrak{M}, w \Vdash \chi$.
9. $\varphi \equiv \Box\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w' \Vdash \psi$ for all $w' \in W$ with Rww'. nml:syn:trw: defn:sub:mmodels-box
10. $\varphi \equiv \Diamond\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w' \Vdash \psi$ for at least one $w' \in W$ with Rww'. nml:syn:trw: defn:sub:mmodels-diamond

Note that by clause (9), a fórmula $\Box\psi$ is true at w whenever there are no w' with Rww'. In such a case $\Box\psi$ is *vacuously* true at w. Also, $\Box\psi$ may be satisfied at w even if ψ is not. The truth of ψ at w does not guarantee the truth of $\Diamond\psi$ at w. This holds, however, if Rww, p.ex., if R is reflexive. If there is no w' such that Rww', then $\mathfrak{M}, w \nVdash \Diamond\varphi$, for any φ.

Problema 1.1. Consider the model of Figura 1.1. Which of the following hold?

1. $\mathfrak{M}, w_1 \Vdash q$;
2. $\mathfrak{M}, w_3 \Vdash \neg q$;
3. $\mathfrak{M}, w_1 \Vdash p \lor q$;
4. $\mathfrak{M}, w_1 \Vdash \Box(p \lor q)$;
5. $\mathfrak{M}, w_3 \Vdash \Box q$;
6. $\mathfrak{M}, w_3 \Vdash \Box\bot$;
7. $\mathfrak{M}, w_1 \Vdash \Diamond q$;
8. $\mathfrak{M}, w_1 \Vdash \Box q$;
9. $\mathfrak{M}, w_1 \Vdash \neg\Box\Box\neg q$.

nml:syn:trw:prop:dual **Proposição 1.7.**

1. *$\mathfrak{M}, w \Vdash \Box\varphi$ iff $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$.*

2. *$\mathfrak{M}, w \Vdash \Diamond\varphi$ iff $\mathfrak{M}, w \Vdash \neg\Box\neg\varphi$.*

Prova. 1. $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$ iff $\mathfrak{M}, w \nVdash \Diamond\neg\varphi$ by definition of $\mathfrak{M}, w \Vdash$. $\mathfrak{M}, w \Vdash \Diamond\neg\varphi$ iff for some w' with Rww', $\mathfrak{M}, w' \Vdash \neg\varphi$. Hence, $\mathfrak{M}, w \nVdash \Diamond\neg\varphi$ iff for all w' with Rww', $\mathfrak{M}, w' \nVdash \neg\varphi$. We also have $\mathfrak{M}, w' \nVdash \neg\varphi$ iff $\mathfrak{M}, w' \Vdash \varphi$. Together we have $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$ iff for all w' with Rww', $\mathfrak{M}, w' \Vdash \varphi$. Again by definition of $\mathfrak{M}, w \Vdash$, that is the case iff $\mathfrak{M}, w \Vdash \Box\varphi$.

2. $\mathfrak{M}, w \Vdash \neg\Box\neg\varphi$ iff $\mathfrak{M} \nVdash \Box\neg\varphi$. $\mathfrak{M}, w \Vdash \Box\neg\varphi$ iff for all w' with Rww', $\mathfrak{M}, w' \Vdash \neg\varphi$. Hence, $\mathfrak{M}, w \nVdash \Box\neg\varphi$ iff for some w' with Rww', $\mathfrak{M}, w' \nVdash \neg\varphi$. We also have $\mathfrak{M}, w' \nVdash \neg\varphi$ iff $\mathfrak{M}, w' \Vdash \varphi$. Together we have $\mathfrak{M}, w \Vdash \neg\Box\neg\varphi$ iff for some w' with Rww', $\mathfrak{M}, w' \Vdash \varphi$. Again by definition of $\mathfrak{M}, w \Vdash$, that is the case iff $\mathfrak{M}, w \Vdash \Diamond\varphi$. □

Problema 1.2. Complete the proof of Proposição 1.7.

Problema 1.3. Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model, and suppose $w_1, w_2 \in W$ are such that:

1. $w_1 \in V(p)$ if and only if $w_2 \in V(p)$ (for every variável proposicional p); and

2. for all $w \in W$: Rw_1w if and only if Rw_2w.

Using induction on fórmulas, show that for all fórmulas φ: $\mathfrak{M}, w_1 \Vdash \varphi$ if and only if $\mathfrak{M}, w_2 \Vdash \varphi$.

Problema 1.4. Let $\mathfrak{M} = \langle W, R, V\rangle$. Show that $\mathfrak{M}, w \Vdash \neg\Diamond\varphi$ if and only if $\mathfrak{M}, w \Vdash \Box\neg\varphi$.

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1.6 Truth in a Model

nml:syn:tru:sec Sometimes we are interested in which fórmulas are true at every world in a given model. Let's introduce a notation for this.

Definição 1.8. A fórmula φ is *true in a model* $M = \langle W, R, V\rangle$, written $\mathfrak{M} \Vdash \varphi$, if and only if $\mathfrak{M}, w \Vdash \varphi$ for every $w \in W$.

nml:syn:tru:prop:truthfacts **Proposição 1.9.**

1. *If $\mathfrak{M} \Vdash \varphi$ then $\mathfrak{M} \nVdash \neg\varphi$, but* not *vice-versa.*

2. *If $\mathfrak{M} \Vdash \varphi \to \psi$ then $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but* not *vice-versa.*

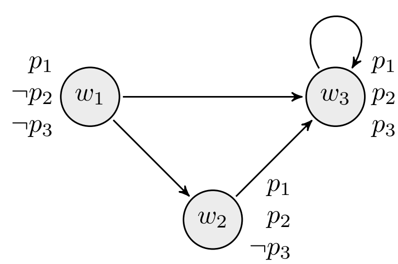
Prova. 1. If $\mathfrak{M} \Vdash \varphi$ then φ is true at all worlds in W, and since $W \neq \emptyset$, it can't be that $\mathfrak{M} \Vdash \neg\varphi$, or else φ would have to be both true and false at some world.

On the other hand, if $\mathfrak{M} \nVdash \neg\varphi$ then φ is true at some world $w \in W$. It does not follow that $\mathfrak{M}, w \Vdash \varphi$ for *every* $w \in W$. For instance, in the model of Figura 1.1, $\mathfrak{M} \nVdash \neg p$, and also $\mathfrak{M} \nVdash p$.

2. Assume $\mathfrak{M} \Vdash \varphi \to \psi$ and $\mathfrak{M} \Vdash \varphi$; to show $\mathfrak{M} \Vdash \psi$ let $w \in W$ be an arbitrary world. Then $\mathfrak{M}, w \Vdash \varphi \to \psi$ and $\mathfrak{M}, w \Vdash \varphi$, so $\mathfrak{M}, w \Vdash \psi$, and since w was arbitrary, $\mathfrak{M} \Vdash \psi$.

To show that the converse fails, we need to find a model $\mathfrak{M}$ such that $\mathfrak{M} \Vdash \varphi$ only if $\mathfrak{M} \Vdash \psi$, but $\mathfrak{M} \nVdash \varphi \to \psi$. Consider again the model of Figura 1.1: $\mathfrak{M} \nVdash p$ and hence (vacuously) $\mathfrak{M} \Vdash p$ only if $\mathfrak{M} \Vdash q$. However, $\mathfrak{M} \nVdash p \to q$, as p is true but q false at w_1. □

Problema 1.5. Consider the following model $\mathfrak{M}$ for the language comprising p_1, p_2, p_3 as the only variáveis proposicionais:



Are the following fórmulas and schemas true in the model $\mathfrak{M}$, O.s., true at every world in $\mathfrak{M}$? Explain.

1. $p \to \Diamond p$ (for p atomic);
2. $\varphi \to \Diamond\varphi$ (for φ arbitrary);
3. $\Box p \to p$ (for p atomic);
4. $\neg p \to \Diamond\Box p$ (for p atomic);
5. $\Diamond\Box\varphi$ (for φ arbitrary);
6. $\Box\Diamond p$ (for p atomic).

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1.7 Validity

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Fórmulas that are true in all models, O.s., true at every world in every model, are particularly interesting. They represent those modal propositions which are true regardless of how $\Box$ and $\Diamond$ are interpreted, as long as the interpretation is "normal" in the sense that it is generated by some accessibility relation on possible worlds. We call such fórmulas *valid.* For instance, $\Box(p \land q) \to \Box p$ is valid. Some fórmulas one might expect to be valid on the basis of the alethic interpretation of $\Box$, such as $\Box p \to p$, are not valid, however. Part of the interest of relational models is that different interpretations of $\Box$ and $\Diamond$ can be captured by different kinds of accessibility relations. This suggests that we should define validity not just relative to *all* models, but relative to all models *of a certain kind*. It will turn out, p.ex., that $\Box p \to p$ is true in all models where every world is accessible from itself, O.s., R is reflexive. Defining validity relative to classes of models enables us to formulate this succinctly: $\Box p \to p$ is valid in the class of reflexive models. explanation

Definição 1.10. A fórmula φ is *valid* in a class $\mathcal{C}$ of models if it is true in every model in $\mathcal{C}$ (O.s., true at every world in every model in $\mathcal{C}$). If φ is valid in $\mathcal{C}$, we write $\mathcal{C} \vDash \varphi$, and we write $\vDash \varphi$ if φ is valid in the class of *all* models.

nml:syn:val: prop:subset-class

Proposição 1.11. *If φ is valid in $\mathcal{C}$ it is also valid in each class $\mathcal{C}' \subseteq \mathcal{C}$.*

nml:syn:val: prop:Nec-rule

Proposição 1.12. *If φ is valid, then so is $\Box\varphi$.*

Prova. Assume $\vDash \varphi$. To show $\vDash \Box\varphi$ let $\mathfrak{M} = \langle W, R, V \rangle$ be a model and $w \in W$. If Rww' then $\mathfrak{M}, w' \Vdash \varphi$, since φ is valid, and so also $\mathfrak{M}, w \Vdash \Box\varphi$. Since $\mathfrak{M}$ and w were arbitrary, $\vDash \Box\varphi$. $\Box$

Problema 1.6. Show that the following are valid:

1. $\Box p \to \Box(q \to p)$;
2. $\Box\neg\bot$;
3. $\Box p \to (\Box q \to \Box p)$.

Problema 1.7. Show that $\varphi \to \Box\varphi$ is valid in the class $\mathcal{C}$ of models $\mathfrak{M} = \langle W, R, V \rangle$ where $W = \{w\}$. Similarly, show that $\psi \to \Box\varphi$ and $\Diamond\varphi \to \psi$ are valid in the class of models $\mathfrak{M} = \langle W, R, V \rangle$ where $R = \emptyset$.

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1.8 Tautological Instances

explanation A modal-free formula is a tautology if it is true under every truth-value assignment. Clearly, every tautology is true at every world in every model. But for fórmulas involving $\Box$ and $\Diamond$, the notion of tautology is not defined. Is it the case, p.ex., that $\Box p \lor \neg\Box p$—an instance of the principle of excluded middle—is valid? The notion of a *tautological instance* helps: a fórmula that is a substitution instance of a (non-modal) tautology. It is not surprising, but still requires proof, that every tautological instance is valid. nml:syn:tau:sec

Definição 1.13. A modal fórmula ψ is a *tautological instance* if and only if there is a modal-free tautology φ with variáveis proposicionais $p_1, \ldots, p_n$ and fórmulas $\theta_1, \ldots, \theta_n$ such that $\psi \equiv \varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$.

Lema 1.14. *Suppose φ is a modal-free fórmula whose variáveis proposicionais are $p_1, \ldots, p_n$, and let $\theta_1, \ldots, \theta_n$ be modal fórmulas. Then for any assignment $\mathfrak{v}$, any model $\mathfrak{M} = \langle W, R, V\rangle$, and any $w \in W$ such that $\mathfrak{v}(p_i) = \mathbb{T}$ if and only if $\mathfrak{M}, w \Vdash \theta_i$ we have that $\mathfrak{v} \vDash \varphi$ if and only if $\mathfrak{M}, w \Vdash \varphi[\theta_1/p_1, \ldots, \theta_n/p_n]$.* *nml:syn:tau:lem:valid-taut*

Prova. By induction on φ.

1. $\varphi \equiv \bot$: Both $\mathfrak{v} \nvDash \bot$ and $\mathfrak{M}, w \nVdash \bot$.

2. $\varphi \equiv \top$: Both $\mathfrak{v} \vDash \top$ and $\mathfrak{M}, w \Vdash \top$.

3. $\varphi \equiv p_i$:

$$\begin{aligned}
\mathfrak{v} \vDash p_i \Leftrightarrow{}& \mathfrak{v}(p_i) = \mathbb{T} \\
&\text{by definition of } \mathfrak{v} \vDash p_i \\
\Leftrightarrow{}& \mathfrak{M}, w \Vdash \theta_i \\
&\text{by assumption} \\
\Leftrightarrow{}& \mathfrak{M}, w \Vdash p_i[\theta_1/p_1, \ldots, \theta_n/p_n] \\
&\text{since } p_i[\theta_1/p_1, \ldots, \theta_n/p_n] \equiv \theta_i.
\end{aligned}$$

4. $\varphi \equiv \neg\psi$:

$$\begin{aligned}
\mathfrak{v} \vDash \neg\psi \Leftrightarrow{}& \mathfrak{v} \nvDash \psi \\
&\text{by definition of } \mathfrak{v} \vDash; \\
\Leftrightarrow{}& \mathfrak{M}, w \nVdash \psi[\theta_1/p_1, \ldots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
\Leftrightarrow{}& \mathfrak{M}, w \Vdash \neg\psi[\theta_1/p_1, \ldots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{v} \vDash.
\end{aligned}$$

5. $\varphi \equiv (\psi \land \chi)$:

$$\begin{aligned}
\mathfrak{v} \vDash \psi \land \chi &\Leftrightarrow \mathfrak{v} \vDash \psi \text{ and } \mathfrak{v} \vDash \chi \\
&\qquad \text{by definition of } \mathfrak{v} \vDash \\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad\ \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\qquad \text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \land \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\qquad \text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

6. $\varphi \equiv (\psi \lor \chi)$:

$$\begin{aligned}
\mathfrak{v} \vDash \psi \lor \chi &\Leftrightarrow \mathfrak{v} \vDash \psi \text{ or } \mathfrak{v} \vDash \chi \\
&\qquad \text{by definition of } \mathfrak{v} \vDash; \\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} \\
&\quad\ \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\qquad \text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \lor \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\qquad \text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

7. $\varphi \equiv (\psi \rightarrow \chi)$:

$$\begin{aligned}
\mathfrak{v} \vDash \psi \rightarrow \chi &\Leftrightarrow \mathfrak{v} \nvDash \psi \text{ or } \mathfrak{v} \vDash \chi \\
&\qquad \text{by definition of } \mathfrak{v} \vDash \\
&\Leftrightarrow \mathfrak{M}, w \nVdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} \\
&\quad\ \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\qquad \text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \rightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\qquad \text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

8. $\varphi \equiv (\psi \leftrightarrow \chi)$:

$$\begin{aligned}
\mathfrak{v} \vDash \psi \rightarrow \chi &\Leftrightarrow \text{either } \mathfrak{v} \vDash \psi \text{ and } \mathfrak{v} \vDash \chi \\
&\quad\ \text{or } \mathfrak{v} \nvDash \psi \text{ and } \mathfrak{v} \nvDash \chi \\
&\qquad \text{by definition of } \mathfrak{v} \vDash \\
&\Leftrightarrow \text{either } \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\qquad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\quad\ \text{or } \mathfrak{M}, w \nVdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\qquad \mathfrak{M}, w \nVdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\qquad \text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \leftrightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\qquad \text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

□

Proposição 1.15. *All tautological instances are valid.* *nml:syn:tau: prop:valid-taut*

Prova. Contrapositively, suppose φ is such that $\mathfrak{M}, w \nVdash \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$, for some model $\mathfrak{M}$ and world w. Define an assignment $\mathfrak{v}$ such that $\mathfrak{v}(p_i) = \mathbb{T}$ if and only if $\mathfrak{M}, w \Vdash \theta_i$ (and $\mathfrak{v}$ assigns arbitrary values to $q \notin \{p_1, \dots, p_n\}$). Then by Lema 1.14, $\mathfrak{v} \nvDash \varphi$, so φ is not a tautology. □

-NoValue-

1.9 Schemas and Validity

nml:syn:sch: sec

Definição 1.16. A *schema* is a set of fórmulas comprising all and only the substitution instances of some modal fórmula χ, O.s.,

$$\{\psi : \exists\theta_1, \dots, \exists\theta_n\,(\psi = \chi[\theta_1/p_1, \dots, \theta_n/p_n])\}.$$

The fórmula χ is called the *characteristic* fórmula of the schema, and it is unique up to a renaming of the variáveis proposicionais. A fórmula φ is an *instance* of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting 'φ', 'ψ', …, for the atomic components of χ. So, for instance, the following denote schemas: 'φ', '$\varphi \to \Box\varphi$', '$\varphi \to (\psi \to \varphi)$'. They correspond to the characteristic fórmulas p, $p \to \Box p$, $p \to (q \to p)$. The schema 'φ' denotes the set of *all* fórmulas.

Definição 1.17. A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

Proposição 1.18. *The following schema* K *is valid* *nml:syn:sch: prop:Kvalid*

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi). \tag{K}$$

Prova. We need to show that all instances of the schema are true at every world in every model. So let $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that the consequent is true as well. In this case, let $\mathfrak{M}, w \Vdash \Box(\varphi \to \psi)$ and $\mathfrak{M}, w \Vdash \Box\varphi$. We need to show $\mathfrak{M}, w \Vdash \Box\psi$. So let w' be arbitrary such that Rww'. Then by the first assumption $\mathfrak{M}, w' \Vdash \varphi \to \psi$ and by the second assumption $\mathfrak{M}, w' \Vdash \varphi$. It follows that $\mathfrak{M}, w' \Vdash \psi$. Since w' was arbitrary, $\mathfrak{M}, w \Vdash \Box\psi$. □

Proposição 1.19. *The following schema* DUAL *is valid* *nml:syn:sch: prop:Dual-valid*

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi. \tag{DUAL}$$

Prova. Exercise. □

Problema 1.8. Prove Proposição 1.19.

nml:syn:sch: prop:soundMP **Proposição 1.20.** *If φ and $\varphi \to \psi$ are true at a world in a model then so is ψ. Hence, the valid fórmulas are closed under modus ponens.*

nml:syn:sch: prop:valid-instances **Proposição 1.21.** *A fórmula φ is valid iff all its substitution instances are. In other words, a schema is valid iff its characteristic fórmula is.*

Prova. The "if" direction is obvious, since φ is a substitution instance of itself.

To prove the "only if" direction, we show the following: Suppose $\mathfrak{M} = \langle W, R, V \rangle$ is a modal model, and $\psi \equiv \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is a substitution instance of φ. Define $\mathfrak{M}' = \langle W, R, V' \rangle$ by $V'(p_i) = \{w : \mathfrak{M}, w \Vdash \theta_i\}$. Then $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}', w \Vdash \varphi$, for any $w \in W$. (We leave the proof as an exercise.) Now suppose that φ was valid, but some substitution instance ψ of φ was not valid. Then for some $\mathfrak{M} = \langle W, R, V \rangle$ and some $w \in W$, $\mathfrak{M}, w \nVdash \psi$. But then $\mathfrak{M}', w \nVdash \varphi$ by the claim, and φ is not valid, a contradiction. □

Problema 1.9. Prove the claim in the "only if" part of the proof of Proposição 1.21. (Hint: use induction on φ.)

Note, however, that it is not true that a schema is true in a model iff its characteristic formula is. Of course, the "only if" direction holds: if every instance of φ is true in $\mathfrak{M}$, φ itself is true in $\mathfrak{M}$. But it may happen that φ is true in $\mathfrak{M}$ but some instance of φ is false at some world in $\mathfrak{M}$. For a very simple counterexample consider p in a model with only one world w and $V(p) = \{w\}$, so that p is true at w. But $\bot$ is an instance of p, and not true at w.

Problema 1.10. Show that none of the following fórmulas are valid:

D: $\Box p \to \Diamond p$;

T: $\Box p \to p$;

B: $p \to \Box \Diamond p$;

4: $\Box p \to \Box\Box p$;

5: $\Diamond p \to \Box \Diamond p$.

Problema 1.11. Prove that the schemas in the first column of Tabela 1.1 are valid and those in the second column are not valid.

Problema 1.12. Decide whether the following schemas are valid or invalid:

1. $(\Diamond \varphi \to \Box \psi) \to (\Box \varphi \to \Box \psi)$;
2. $\Diamond(\varphi \to \psi) \lor \Box(\psi \to \varphi)$.

Problema 1.13. For each of the following schemas find a model $\mathfrak{M}$ such that every instance of the fórmula is true in $\mathfrak{M}$:

Valid Schemas	*Invalid Schemas*
$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	$\Box(\varphi \lor \psi) \to (\Box\varphi \lor \Box\psi)$
$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$	$(\Diamond\varphi \land \Diamond\psi) \to \Diamond(\varphi \land \psi)$
$\Box(\varphi \land \psi) \leftrightarrow (\Box\varphi \land \Box\psi)$	$\varphi \to \Box\varphi$
$\Box\varphi \to \Box(\psi \to \varphi)$	$\Box\Diamond\varphi \to \psi$
$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$	$\Box\Box\varphi \to \Box\varphi$
$\Diamond(\varphi \lor \psi) \leftrightarrow (\Diamond\varphi \lor \Diamond\psi)$	$\Box\Diamond\varphi \to \Diamond\Box\varphi$.

Tabela 1.1: Valid and (or?) invalid schemas.

nml:syn:sch:
tab:valid-invalidSchemas

1. $p \to \Diamond\Diamond p$;

2. $\Diamond p \to \Box p$.

-NoValue-

1.10 Entailment

explanation With the definition of truth at a world, we can define an entailment relation between fórmulas. A fórmula ψ entails φ iff, whenever ψ is true, φ is true as well. Here, "whenever" means both "whichever model we consider" as well as "whichever world in that model we consider." nml:syn:ent:sec

Definição 1.22. If Γ is a set of fórmulas and φ a fórmula, then Γ *entails* φ, in symbols: $\Gamma \vDash \varphi$, if and only if for every model $\mathfrak{M} = \langle W, R, V\rangle$ and world $w \in W$, if $\mathfrak{M}, w \Vdash \psi$ for every $\psi \in \Gamma$, then $\mathfrak{M}, w \Vdash \varphi$. If Γ contains a single fórmula ψ, then we write $\psi \vDash \varphi$.

Exemplo 1.23. To show that a fórmula entails another, we have to reason about all models, using the definition of $\mathfrak{M}, w \Vdash$. For instance, to show $p \to \Diamond p \vDash \Box\neg p \to \neg p$, we might argue as follows: Consider a model $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$, and suppose $\mathfrak{M}, w \Vdash p \to \Diamond p$. We have to show that $\mathfrak{M}, w \Vdash \Box\neg p \to \neg p$. Suppose not. Then $\mathfrak{M}, w \Vdash \Box\neg p$ and $\mathfrak{M}, w \nVdash \neg p$. Since $\mathfrak{M}, w \nVdash \neg p$, $\mathfrak{M}, w \Vdash p$. By assumption, $\mathfrak{M}, w \Vdash p \to \Diamond p$, hence $\mathfrak{M}, w \Vdash \Diamond p$. By definition of $\mathfrak{M}, w \Vdash \Diamond p$, there is some w' with Rww' such that $\mathfrak{M}, w' \Vdash p$. Since also $\mathfrak{M}, w \Vdash \Box\neg p$, $\mathfrak{M}, w' \Vdash \neg p$, a contradiction.

To show that a fórmula ψ does not entail another φ, we have to give a counterexample, O.s., a model $\mathfrak{M} = \langle W, R, V\rangle$ where we show that at some world $w \in W$, $\mathfrak{M}, w \Vdash \psi$ but $\mathfrak{M}, w \nVdash \varphi$. Let's show that $p \to \Diamond p \nvDash \Box p \to p$. Consider the model in Figura 1.2. We have $\mathfrak{M}, w_1 \Vdash \Diamond p$ and hence $\mathfrak{M}, w_1 \Vdash p \to \Diamond p$. However, since $\mathfrak{M}, w_1 \Vdash \Box p$ but $\mathfrak{M}, w_1 \nVdash p$, we have $\mathfrak{M}, w_1 \nVdash \Box p \to p$.

Often very simple counterexamples suffice. The model $\mathfrak{M}' = \{W', R', V'\}$ with $W' = \{w\}$, $R' = \emptyset$, and $V'(p) = \emptyset$ is also a counterexample: Since $\mathfrak{M}', w \nVdash p$, $\mathfrak{M}', w \Vdash p \to \Diamond p$. As no worlds are accessible from w, we have $\mathfrak{M}', w \Vdash \Box p$, and so $\mathfrak{M}', w \nVdash \Box p \to p$.

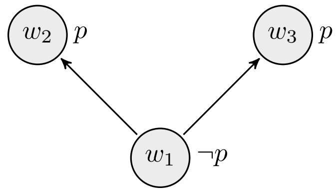

Figura 1.2: Counterexample to $p \to \Diamond p \vDash \Box p \to p$.

nml:syn:ent:
fig:counterex

Problema 1.14. Show that $\Box(\varphi \land \psi) \vDash \Box\varphi$.

Problema 1.15. Show that $\Box(p \to q) \nvDash p \to \Box q$ and $p \to \Box q \nvDash \Box(p \to q)$.

Capítulo 2

Frame Definability

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2.1 Introduction

One question that interests modal logicians is the relationship between the accessibility relation and the truth of certain fórmulas in models with that accessibility relation. For instance, suppose the accessibility relation is reflexive, O.s., for every $w \in W$, Rww. In other words, every world is accessible from itself. That means that when $\Box\varphi$ is true at a world w, w itself is among the accessible worlds at which φ must therefore be true. So, if the accessibility relation R of $\mathfrak{M}$ is reflexive, then whatever world w and formula φ we take, $\Box\varphi \to \varphi$ will be true there (in other words, the schema $\Box p \to p$ and all its substitution instances are true in $\mathfrak{M}$). nml:frd:int:sec

The converse, however, is false. It's not the case, p.ex., that if $\Box p \to p$ is true in $\mathfrak{M}$, then R is reflexive. For we can easily find a non-reflexive model $\mathfrak{M}$ where $\Box p \to p$ is true at all worlds: take the model with a single world w, not accessible from itself, but with $w \in V(p)$. By picking the truth value of p suitably, we can make $\Box\varphi \to \varphi$ true in a model that is not reflexive.

The solution is to remove the variable assignment V from the equation. If we require that $\Box p \to p$ is true at all worlds in $\mathfrak{M}$, regardless of which worlds are in $V(p)$, then it is necessary that R is reflexive. For in any non-reflexive model, there will be at least one world w such that not Rww. If we set $V(p) = W \setminus \{w\}$, then p will be true at all worlds other than w, and so at all worlds accessible from w (since w is guaranteed not to be accessible from w, and w is the only world where p is false). On the other hand, p is false at w, so $\Box p \to p$ is false at w.

This suggests that we should introduce a notation for model structures without a valuation: we call these *frames*. A frame $\mathfrak{F}$ is simply a pair $\langle W, R\rangle$ consisting of a set of worlds with an accessibility relation. Every model $\langle W, R, V\rangle$ is then, as we say, *based on* the frame $\langle W, R\rangle$. Conversely, a frame determines the class of models based on it; and a class of frames determines the class of

If R is ...	*then ... is true in* $\mathfrak{M}$*:*	
serial: $\forall u \exists v Ruv$	$\Box p \to \Diamond p$	(D)
reflexive: $\forall w Rww$	$\Box p \to p$	(T)
symmetric: $\forall u \forall v (Ruv \to Rvu)$	$p \to \Box\Diamond p$	(B)
transitive: $\forall u \forall v \forall w ((Ruv \land Rvw) \to Ruw)$	$\Box p \to \Box\Box p$	(4)
euclidean: $\forall w \forall u \forall v ((Rwu \land Rwv) \to Ruv)$	$\Diamond p \to \Box\Diamond p$	(5)

Tabela 2.1: Five correspondence facts.

nml:frd:acc:tab:five

models which are based on any frame in the class. And we can define $\mathfrak{F} \vDash \varphi$, the notion of a fórmula being *valid* in a frame as: $\mathfrak{M} \Vdash \varphi$ for all $\mathfrak{M}$ based on $\mathfrak{F}$.

With this notation, we can establish correspondence relations between fórmulas and classes of frames: p.ex., $\mathfrak{F} \vDash \Box p \to p$ if, and only if, $\mathfrak{F}$ is reflexive.

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2.2 Properties of Accessibility Relations

nml:frd:acc:sec

Many modal fórmulas turn out to be characteristic of simple, and even familiar, properties of the accessibility relation. In one direction, that means that any model that has a given property makes a corresponding fórmula (and all its substitution instances) true. We begin with five classical examples of kinds of accessibility relations and the formulas the truth of which they guarantee.

nml:frd:acc:thm:soundschemas

Teorema 2.1. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. If R has the property on the left side of Tabela 2.1, every instance of the fórmula on the right side is true in $\mathfrak{M}$.*

Prova. Here is the case for B: to show that the schema is true in a model we need to show that all of its instances are true at all worlds in the model. So let $\varphi \to \Box\Diamond\varphi$ be a given instance of B, and let $w \in W$ be an arbitrary world. Suppose the antecedent φ is true at w, in order to show that $\Box\Diamond\varphi$ is true at w. So we need to show that $\Diamond\varphi$ is true at all w' accessible from w. Now, for any w' such that Rww' we have, using the hypothesis of symmetry, that also $Rw'w$ (see Figura 2.1). Since $\mathfrak{M}, w \Vdash \varphi$, we have $\mathfrak{M}, w' \Vdash \Diamond\varphi$. Since w' was an arbitrary world such that Rww', we have $\mathfrak{M}, w \Vdash \Box\Diamond\varphi$.

We leave the other cases as exercises. □

Problema 2.1. Complete the proof of Teorema 2.1.

Notice that the converse implications of Teorema 2.1 do not hold: it's not true that if a model verifies a schema, then the accessibility relation of that model has the corresponding property. In the case of T and reflexive models,

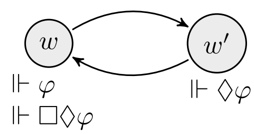

Figura 2.1: The argument from symmetry.

nml:frd:acc: fig:Bsymm

it is easy to give an example of a model in which T itself fails: let $W = \{w\}$ and $V(p) = \emptyset$. Then R is not reflexive, but $\mathfrak{M}, w \Vdash \Box p$ and $\mathfrak{M}, w \nVdash p$. But here we have just a single instance of T that fails in $\mathfrak{M}$; other instances, p.ex., $\Box\neg p \to \neg p$ are true. It is harder to give examples where *every substitution instance* of T is true in $\mathfrak{M}$ and $\mathfrak{M}$ is not reflexive. But there are such models, too:

Proposição 2.2. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model such that $W = \{u, v\}$, where worlds u and v are related by R: O.s., both Ruv and Rvu. Suppose that for all p: $u \in V(p) \Leftrightarrow v \in V(p)$. Then:* nml:frd:acc: prop:reflexive

1. *For all φ: $\mathfrak{M}, u \Vdash \varphi$ if and only if $\mathfrak{M}, v \Vdash \varphi$ (use induction on φ).*

2. *Every instance of* T *is true in $\mathfrak{M}$.*

Since $\mathfrak{M}$ is not reflexive (it is, in fact, irreflexive*), the converse of Teorema 2.1 fails in the case of* T *(similar arguments can be given for some—though not all—the other schemas mentioned in Teorema 2.1).*

Problema 2.2. Prove the claims in Proposição 2.2.

Although we will focus on the five classical fórmulas D, T, B, 4, and 5, we record in Tabela 2.2 a few more properties of accessibility relations. The accessibility relation R is partially functional, if from every world at most one world is accessible. If it is the case that from every world exactly one world is accessible, we call it functional. (Thus the functional relations are precisely those that are both serial and partially functional). They are called "functional" because the accessibility relation operates like a (partial) function. A relation is weakly dense if whenever Ruv, there is a w "between" u and v. So weakly dense relations are in a sense the opposite of transitive relations: in a transitive relation, whenever you can reach v from u by a detour via w, you can reach v from u directly; in a weakly dense relation, whenever you can reach v from u directly, you can also reach it by a detour via some w. A relation is weakly directed if whenever you can reach worlds u and v from some world w, you can reach a single world t from both u and v—this is sometimes called the "diamond property" or "confluence."

Problema 2.3. Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model. Show that if R satisfies the left-hand properties of Tabela 2.2, every instance of the corresponding right-hand formula is true in $\mathfrak{M}$.

If R is ...	*then ... is true in $\mathfrak{M}$:*
partially functional: $\forall w \forall u \forall v((Rwu \land Rwv) \to u = v)$	$\Diamond p \to \Box p$
functional: $\forall w \exists v \forall u(Rwu \leftrightarrow u = v)$	$\Diamond p \leftrightarrow \Box p$
weakly dense: $\forall u \forall v(Ruv \to \exists w(Ruw \land Rwv))$	$\Box\Box p \to \Box p$
weakly connected: $\forall w \forall u \forall v((Rwu \land Rwv) \to (Ruv \lor u = v \lor Rvu))$	$\Box((p \land \Box p) \to q) \lor \Box((q \land \Box q) \to p)$ (L)
weakly directed: $\forall w \forall u \forall v((Rwu \land Rwv) \to \exists t(Rut \land Rvt))$	$\Diamond\Box p \to \Box\Diamond p$ (G)

Tabela 2.2: Five more correspondence facts.

nml:frd:acc:
tab:anotherfive

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2.3 Frames

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sec

Definição 2.3. A *frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a non-empty set of worlds and R a binary relation on W. A model $\mathfrak{M}$ is *based on* a frame $\mathfrak{F} = \langle W, R\rangle$ if and only if $\mathfrak{M} = \langle W, R, V\rangle$ for some valuation V.

Definição 2.4. If $\mathfrak{F}$ is a frame, we say that φ is *valid in $\mathfrak{F}$*, $\mathfrak{F} \vDash \varphi$, if $\mathfrak{M} \Vdash \varphi$ for every model $\mathfrak{M}$ based on $\mathfrak{F}$.

If $\mathcal{F}$ is a class of frames, we say φ is *valid in $\mathcal{F}$*, $\mathcal{F} \vDash \varphi$, iff $\mathfrak{F} \vDash \varphi$ for every frame $\mathfrak{F} \in \mathcal{F}$.

The reason frames are interesting is that correspondence between schemas and properties of the accessibility relation R is at the level of frames, *not of models*. For instance, although T is true in all reflexive models, not every model in which T is true is reflexive. However, it *is* true that not only is T *valid* on all reflexive *frames*, also every frame in which T is valid is reflexive.

Observação 1. Validity in a class of frames is a special case of the notion of validity in a class of models: $\mathcal{F} \vDash \varphi$ iff $\mathcal{C} \vDash \varphi$ where $\mathcal{C}$ is the class of all models based on a frame in $\mathcal{F}$.

Obviously, if a fórmula or a schema is valid, O.s., valid with respect to the class of *all* models, it is also valid with respect to any class $\mathcal{F}$ of frames.

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2.4 Frame Definability

Even though the converse implications of Teorema 2.1 fail, they hold if we replace "model" by "frame": for the properties considered in Teorema 2.1, it *is* true that if a fórmula is valid in a *frame* then the accessibility relation of that frame has the corresponding property. So, the fórmulas considered *define* the classes of frames that have the corresponding property. nml:frd:def:sec

Definição 2.5. If $\mathcal{F}$ is a class of frames, we say φ *defines* $\mathcal{F}$ iff $\mathfrak{F} \vDash \varphi$ for all and only frames $\mathfrak{F} \in \mathcal{F}$.

We now proceed to establish the full definability results for frames.

Teorema 2.6. *If the fórmula on the right side of Tabela 2.1 is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ has the property on the left side.* *nml:frd:def:thm:fullCorrespondence*

Prova.

1. Suppose D is valid in $\mathfrak{F} = \langle W, R \rangle$, O.s., $\mathfrak{F} \vDash \Box p \to \Diamond p$. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model based on $\mathfrak{F}$, and $w \in W$. We have to show that there is a v such that Rwv. Suppose not: then both $\mathfrak{M} \Vdash \Box\varphi$ and $\mathfrak{M}, w \nVdash \Diamond\varphi$ for any φ, including p. But then $\mathfrak{M}, w \nVdash \Box p \to \Diamond p$, contradicting the assumption that $\mathfrak{F} \vDash \Box p \to \Diamond p$.

2. Suppose T is valid in $\mathfrak{F}$, O.s., $\mathfrak{F} \vDash \Box p \to p$. Let $w \in W$ be an arbitrary world; we need to show Rww. Let $u \in V(p)$ if and only if Rwu (when q is other than p, $V(q)$ is arbitrary, say $V(q) = \emptyset$). Let $\mathfrak{M} = \langle W, R, V \rangle$. By construction, for all u such that Rwu: $\mathfrak{M}, u \Vdash p$, and hence $\mathfrak{M}, w \Vdash \Box p$. But by hypothesis $\Box p \to p$ is true at w, so that $\mathfrak{M}, w \Vdash p$, but by definition of V this is possible only if Rww.

3. We prove the contrapositive: Suppose $\mathfrak{F}$ is not symmetric, we show that B, O.s., $p \to \Box\Diamond p$ is not valid in $\mathfrak{F} = \langle W, R \rangle$. If $\mathfrak{F}$ is not symmetric, there are u, $v \in W$ such that Ruv but not Rvu. Define V such that $w \in V(p)$ if and only if not Rvw (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V \rangle$. Now, by definition of V, $\mathfrak{M}, w \Vdash p$ for all w such that not Rvw, in particular, $\mathfrak{M}, u \Vdash p$ since not Rvu. Also, since Rvw iff $w \notin V(p)$, there is no w such that Rvw and $\mathfrak{M}, w \Vdash p$, and hence $\mathfrak{M}, v \nVdash \Diamond p$. Since Ruv, also $\mathfrak{M}, u \nVdash \Box\Diamond p$. It follows that $\mathfrak{M}, u \nVdash p \to \Box\Diamond p$, and so B is not valid in $\mathfrak{F}$.

4. Suppose 4 is valid in $\mathfrak{F} = \langle W, R \rangle$, O.s., $\mathfrak{F} \vDash \Box p \to \Box\Box p$, and let u, v, $w \in W$ be arbitrary worlds such that Ruv and Rvw; we need to show that Ruw. Define V such that $z \in V(p)$ if and only if Ruz (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V \rangle$. By definition of V, $\mathfrak{M}, z \Vdash p$ for all z such that Ruz, and hence $\mathfrak{M}, u \Vdash \Box p$. But by hypothesis 4, $\Box p \to \Box\Box p$, is true at u, so that $\mathfrak{M}, u \Vdash \Box\Box p$. Since Ruv and Rvw, we have $\mathfrak{M}, w \Vdash p$, but by definition of V this is possible only if Ruw, as desired.

 normal-modal-logic rev: 51c2271 (2026-07-12) por OLP /

5. We proceed contrapositively, assuming that the frame $\mathfrak{F} = \langle W, R\rangle$ is not euclidean, and show that it falsifies 5, O.s., $\mathfrak{F} \nVdash \Diamond p \to \Box\Diamond p$. Suppose there are worlds u, v, $w \in W$ such that Rwu and Rwv but not Ruv. Define V such that for all worlds z, $z \in V(p)$ if and only if it is *not* the case that Ruz. Let $\mathfrak{M} = \langle W, R, V\rangle$. Then by hypothesis $\mathfrak{M}, v \Vdash p$ and since Rwv also $\mathfrak{M}, w \Vdash \Diamond p$. However, there is no world y such that Ruy and $\mathfrak{M}, y \Vdash p$ so $\mathfrak{M}, u \nVdash \Diamond p$. Since Rwu, it follows that $\mathfrak{M}, w \nVdash \Box\Diamond p$, so that 5, $\Diamond p \to \Box\Diamond p$, fails at w. □

You'll notice a difference between the proof for D and the other cases: no mention was made of the valuation V. In effect, we proved that if $\mathfrak{M} \Vdash$ D then $\mathfrak{M}$ is serial. So D defines the class of serial *models*, not just frames.

nml:frd:def: prop:D-serial

Corolário 2.7. *Any model where* D *is true is serial.*

Corolário 2.8. *Each fórmula on the right side of Tabela 2.1 defines the class of frames which have the property on the left side.*

Prova. In Teorema 2.1, we proved that if a model has the property on the left, the fórmula on the right is true in it. Thus, if a frame $\mathfrak{F}$ has the property on the left, the fórmula on the right is valid in $\mathfrak{F}$. In Teorema 2.6, we proved the converse implications: if a fórmula on the right is valid in $\mathfrak{F}$, $\mathfrak{F}$ has the property on the left. □

Problema 2.4. Show that if the fórmula on the right side of Tabela 2.2 is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ has the property on the left side. To do this, consider a frame that does *not* satisfy the property on the left, and define a suitable V such that the fórmula on the right is false at some world.

Teorema 2.6 also shows that the properties can be combined: for instance if both B and 4 are valid in $\mathfrak{F}$ then the frame is both symmetric and transitive, etc. Many important modal logics are characterized as the set of fórmulas valid in all frames that combine some frame properties, and so we can characterize them as the set of fórmulas valid in all frames in which the corresponding defining fórmulas are valid. For instance, the classical system **S4** is the set of all fórmulas valid in all reflexive and transitive frames, O.s., in all those where both T and 4 are valid. **S5** is the set of all formulas valid in all reflexive, symmetric, and euclidean frames, O.s., all those where all of T, B, and 5 are valid.

Logical relationships between properties of R in general correspond to relationships between the corresponding defining fórmulas. For instance, every reflexive relation is serial; hence, whenever T is valid in a frame, so is D. (Note that this relationship is *not* that of entailment. It is not the case that whenever $\mathfrak{M}, w \Vdash$ T then $\mathfrak{M}, w \Vdash$ D.) We record some such relationships.

nml:frd:def: prop:relation-facts

Proposição 2.9. *Let R be a binary relation on a set W; then:*

1. *If R is reflexive, then it is serial.*

2. *If R is symmetric, then it is transitive if and only if it is euclidean.*

3. *If R is symmetric or euclidean then it is weakly directed (it has the "diamond property").*

4. *If R is euclidean then it is weakly connected.*

5. *If R is functional then it is serial.*

Problema 2.5. Prove Proposição 2.9.

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2.5 First-order Definability

We've seen that a number of properties of accessibility relations of frames can be defined by modal fórmulas. For instance, symmetry of frames can be defined by the fórmula B, $p \to \Box\Diamond p$. The conditions we've encountered so far can all be expressed by first-order fórmulas in a linguagem involving a single two-place símbolo de predicado. For instance, symmetry is defined by $\forall x\, \forall y\, (Q(x, y) \to Q(y, x))$ in the sense that a first-order estrutura $\mathfrak{M}$ with $|\mathfrak{M}| = W$ and $Q^{\mathfrak{M}} = R$ satisfies the preceding formula iff R is symmetric. This suggests the following definition: nml:frd:fol:sec

Definição 2.10. A class $\mathcal{F}$ of frames is *first-order definable* if there is a sentença φ in the first-order language with a single two-place símbolo de predicado Q such that $\mathfrak{F} = \langle W, R\rangle \in \mathcal{F}$ iff $\mathfrak{M} \vDash \varphi$ in the first-order estrutura $\mathfrak{M}$ with $|\mathfrak{M}| = W$ and $Q^{\mathfrak{M}} = R$.

It turns out that the properties and modal fórmulas that define them considered so far are exceptional. Not every fórmula defines a first-order definable class of frames, and not every first-order definable class of frames is definable by a modal formula.

A counterexample to the first is given by the Löb formula:

$$\Box(\Box p \to p) \to \Box p. \tag{W}$$

W defines the class of transitive and converse well-founded frames. A relation is well-founded if there is no infinite sequence w_1, w_2, … such that Rw_2w_1, Rw_3w_2, …. For instance, the relation $<$ on $\mathbb{N}$ is well-founded, whereas the relation $<$ on $\mathbb{Z}$ is not. A relation is converse well-founded iff its converse is well-founded. So converse well-founded relations are those where there is no infinite sequence w_1, w_2, … such that Rw_1w_2, Rw_2w_3, ….

There is, however, no first-order fórmula defining transitive converse well-founded relations. For suppose $\mathfrak{M} \vDash \beta$ iff $R = Q^{\mathfrak{M}}$ is transitive converse well-founded. Let φ_n be the fórmula

$$(Q(a_1, a_2) \land \dots \land Q(a_{n-1}, a_n))$$

Now consider the set of fórmulas

$$\Gamma = \{\beta, \varphi_1, \varphi_2, \dots\}.$$

Every finite subset of Γ is satisfiable: Let k be largest such that φ_k is in the subset, $|\mathfrak{M}_k| = \{1, \dots, k\}$, $a_i^{\mathfrak{M}_k} = i$, and $Q^{\mathfrak{M}_k} =<$. Since $<$ on $\{1, \dots, k\}$ is transitive and converse well-founded, $\mathfrak{M}_k \vDash \beta$. $\mathfrak{M}_k \vDash \varphi_i$ by construction, for all $i \leq k$. By the Compactness Theorem for first-order logic, Γ is satisfiable in some estrutura $\mathfrak{M}$. By hypothesis, since $\mathfrak{M} \vDash \beta$, the relation $Q^{\mathfrak{M}}$ is converse well-founded. But clearly, $a_1^{\mathfrak{M}}$, $a_2^{\mathfrak{M}}$, $\dots$ would form an infinite sequence of the kind ruled out by converse well-foundedness.

A counterexample to the second claim is given by the property of universality: for every u and v, Ruv. Universal frames are first-order definable by the formula $\forall x \, \forall y \, Q(x, y)$. However, no modal formula is valid in all and only the universal frames. This is a consequence of a result that is independently interesting: the formulas valid in universal frames are exactly the same as those valid in reflexive, symmetric, and transitive frames. There are reflexive, symmetric, and transitive frames that are not universal, hence every fórmula valid in all universal frames is also valid in some non-universal frames.

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2.6 Equivalence Relations and S5

nml:frd:es5:sec The modal logic **S5** is characterized as the set of fórmulas valid on all universal frames, O.s., every world is accessible from every world, including itself. In such a scenario, $\Box$ corresponds to necessity and $\Diamond$ to possibility: $\Box\varphi$ is true if φ is true at *every* world, and $\Diamond\varphi$ is true if φ is true at *some* world. It turns out that **S5** can also be characterized as the fórmulas valid on all reflexive, symmetric, and transitive frames, O.s., on all *equivalence relations*.

Definição 2.11. A binary relation R on W is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation R on W is *universal* if and only if Ruv for all $u, v \in W$.

Since T, B, and 4 characterize the reflexive, symmetric, and transitive frames, the frames where the accessibility relation is an equivalence relation are exactly those in which all three fórmulas are valid. It turns out that the equivalence relations can also be characterized by other combinations of fórmulas, since the conditions with which we've defined equivalence relations are equivalent to combinations of other familiar conditions on R.

nml:frd:es5:prop:equivalences **Proposição 2.12.** *The following are equivalent:*

1. *R is an equivalence relation;*
2. *R is reflexive and euclidean;*
3. *R is serial, symmetric, and euclidean;*

4. *R is serial, symmetric, and transitive.*

Prova. Exercise. □

Problema 2.6. Prove Proposição 2.12 by showing:

1. If R is symmetric and transitive, it is euclidean.
2. If R is reflexive, it is serial.
3. If R is reflexive and euclidean, it is symmetric.
4. If R is symmetric and euclidean, it is transitive.
5. If R is serial, symmetric, and transitive, it is reflexive.

Explain why this suffices for the proof that the conditions are equivalent.

Proposição 2.12 is the semantic counterpart to Proposição 3.30, in that it gives an equivalent characterization of the modal logic of frames over which R is an equivalence relation (the logic traditionally referred to as **S5**).

What is the relationship between universal and equivalence relations? Although every universal relation is an equivalence relation, clearly not every equivalence relation is universal. However, the fórmulas valid on all universal relations are exactly the same as those valid on all equivalence relations.

Proposição 2.13. *Let R be an equivalence relation, and for each $w \in W$ define the* equivalence class *of w as the set $[w] = \{w' \in W : Rww'\}$. Then:*

1. *$w \in [w]$;*
2. *R is universal on each equivalence class $[w]$;*
3. *The collection of equivalence classes partitions W into mutually exclusive and jointly exhaustive subsets.*

Proposição 2.14. *A fórmula φ is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is an equivalence relation, if and only if it is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is universal. Hence, the logic of universal frames is just* **S5**. nml:frd:es5:prop:S5=univ

Prova. It's immediate to verify that a universal relation R on W is an equivalence. Hence, if φ is valid in all frames where R is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose ψ is a fórmula that fails at a world w in a model $\mathfrak{M} = \langle W, R, V\rangle$ based on a frame $\langle W, R\rangle$, where R is an equivalence on W. So $\mathfrak{M}, w \nVdash \psi$. Define a model $\mathfrak{M}' = \langle W', R', V'\rangle$ as follows:

1. $W' = [w]$;
2. R' is universal on W';

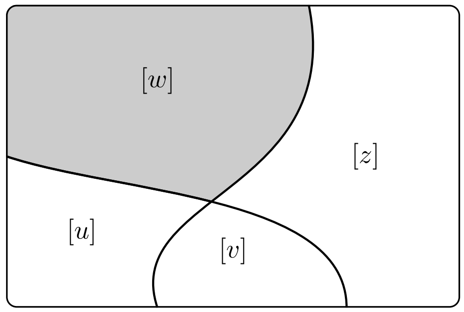

Figura 2.2: A partition of W in equivalence classes.

nml:frd:es5: fig:partition

3. $V'(p) = V(p) \cap W'$.

(So the set W' of worlds in $\mathfrak{M}'$ is represented by the shaded area in Figura 2.2.) It is easy to see that R and R' agree on W'. Then one can show by induction on fórmulas that for all $w' \in W'$: $\mathfrak{M}', w' \Vdash \varphi$ if and only if $\mathfrak{M}, w' \Vdash \varphi$ for each φ (this makes sense since $W' \subseteq W$). In particular, $\mathfrak{M}', w \nVdash \psi$, and ψ fails in a model based on a universal frame. □

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2.7 Second-order Definability

nml:frd:st: sec

Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (O.s., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal fórmula is true at a world are related to first-order fórmulas. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place símbolo de predicado Q for the accessibility relation, but also a one-place símbolo de predicado P_i for the variáveis proposicionais p_i occurring in φ.

Definição 2.15. The *standard translation* $\mathrm{ST}_x(\varphi)$ is inductively defined as follows:

1. $\varphi \equiv \bot$: $\mathrm{ST}_x(\varphi) = \bot$.
2. $\varphi \equiv \bot$: $\mathrm{ST}_x(\varphi) = \top$.
3. $\varphi \equiv p_i$: $\mathrm{ST}_x(\varphi) = P_i(x)$.
4. $\varphi \equiv \neg\psi$: $\mathrm{ST}_x(\varphi) = \neg\mathrm{ST}_x(\psi)$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \wedge \mathrm{ST}_x(\chi))$.

6. $\varphi \equiv (\psi \vee \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \vee \mathrm{ST}_x(\chi))$.
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \rightarrow \mathrm{ST}_x(\chi))$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathrm{ST}_x(\varphi) = (\mathrm{ST}_x(\psi) \leftrightarrow \mathrm{ST}_x(\chi))$.
9. $\varphi \equiv \Box\psi$: $\mathrm{ST}_x(\varphi) = \forall y\,(Q(x,y) \rightarrow \mathrm{ST}_y(\psi))$.
10. $\varphi \equiv \Diamond\psi$: $\mathrm{ST}_x(\varphi) = \exists y\,(Q(x,y) \wedge \mathrm{ST}_y(\psi))$.

For instance, $\mathrm{ST}_x(\Box p \rightarrow p)$ is $\forall y\,(Q(x,y) \rightarrow P(y)) \rightarrow P(x)$. Any estrutura for the language of $\mathrm{ST}_x(\varphi)$ requires a domain, a two-place relation assigned to Q, and subsets of the domain assigned to the one-place símbolos de predicado P_i. In other words, the components of such a estrutura are exactly those of a model for φ: the domain is the set of worlds, the two-place relation assigned to Q is the accessibility relation, and the subsets assigned to P_i are just the assignments $V(p_i)$. It won't surprise that satisfaction of φ in a modal model and of $\mathrm{ST}_x(\varphi)$ in the corresponding estrutura agree:

Proposição 2.16. *Let $\mathfrak{M} = \langle W, R, V\rangle$, $\mathfrak{M}'$ be the first-order estrutura with $|\mathfrak{M}'| = W$, $Q^{\mathfrak{M}'} = R$, and $P_i^{\mathfrak{M}'} = V(p_i)$, and $s(x) = w$. Then* nml:frd:st:prop:st

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}', s \vDash \mathrm{ST}_x(\varphi)$$

Prova. By induction on φ. □

Proposição 2.17. *Suppose φ is a modal fórmula and $\mathfrak{F} = \langle W, R\rangle$ is a frame. Let $\mathfrak{F}'$ be the first-order estrutura with $|\mathfrak{F}'| = W$ and $Q^{\mathfrak{F}'} = R$, and let φ' be the second-order formula*

$$\forall X_1 \ldots \forall X_n\, \forall x\, \mathrm{ST}_x(\varphi)[X_1/P_1, \ldots, X_n/P_n],$$

where $P_1, \ldots, P_n$ are all one-place símbolos de predicado in $\mathrm{ST}_x(\varphi)$. Then

$$\mathfrak{F} \vDash \varphi \text{ iff } \mathfrak{F}' \vDash \varphi'$$

Prova. $\mathfrak{F}' \vDash \varphi'$ iff for every estrutura $\mathfrak{M}'$ where $P_i^{\mathfrak{M}'} \subseteq W$ for $i = 1, \ldots, n$, and for every s with $s(x) \in W$, $\mathfrak{M}', s \vDash \mathrm{ST}_x(\varphi)$. By Proposição 2.16, that is the case iff for all models $\mathfrak{M}$ based on $\mathfrak{F}$ and every world $w \in W$, $\mathfrak{M}, w \Vdash \varphi$, O.s., $\mathfrak{F} \vDash \varphi$. □

Definição 2.18. A class $\mathcal{F}$ of frames is *second-order definable* if there is a sentença φ in the second-order language with a single two-place símbolo de predicado P and quantifiers only over monadic set variables such that $\mathfrak{F} = \langle W, R\rangle \in \mathcal{F}$ iff $\mathfrak{M} \vDash \varphi$ in the estrutura $\mathfrak{M}$ with $|\mathfrak{M}| = W$ and $P^{\mathfrak{M}} = R$.

Corolário 2.19. *If a class of frames is definable by a fórmula φ, the corresponding class of accessibility relations is definable by a monadic second-order sentence.*

Prova. The monadic second-order sentence φ' of the preceding proof has the required property. $\square$

As an example, consider again the fórmula $\Box p \to p$. It defines reflexivity. Reflexivity is of course first-order definable by the sentença $\forall x\, Q(x,x)$. But it is also definable by the monadic second-order sentença

$$\forall X\, \forall x\, (\forall y\, (Q(x,y) \to X(y)) \to X(x)).$$

This means, of course, that the two sentences are equivalent. Here's how you might convince yourself of this directly: First suppose the second-order sentence is true in a estrutura $\mathfrak{M}$. Since x and X are universally quantified, the remainder must hold for any $x \in W$ and set $X \subseteq W$, p.ex., the set $\{z : Rxz\}$ where $R = Q^{\mathfrak{M}}$. So, for any s with $s(x) \in W$ and $s(X) = \{z : Rxz\}$ we have $\mathfrak{M} \vDash \forall y\, (Q(x,y) \to X(y)) \to X(x)$. But by the way we've picked $s(X)$ that means $\mathfrak{M}, s \vDash \forall y\, (Q(x,y) \to Q(x,y)) \to Q(x,x)$, which is equivalent to $Q(x,x)$ since the antecedent is valid. Since $s(x)$ is arbitrary, we have $\mathfrak{M} \vDash \forall x\, Q(x,x)$.

Now suppose that $\mathfrak{M} \vDash \forall x\, Q(x,x)$ and show that $\mathfrak{M} \vDash \forall X\, \forall x\, (\forall y\, (Q(x,y) \to X(y)) \to X(x))$. Pick any assignment s, and assume $\mathfrak{M}, s \vDash \forall y\, (Q(x,y) \to X(y))$. Let s' be the y-variant of s with $s'(y) = s(x)$; we have $\mathfrak{M}, s' \vDash Q(x,y) \to X(y)$, O.s., $\mathfrak{M}, s \vDash Q(x,x) \to X(x)$. Since $\mathfrak{M} \vDash \forall x\, Q(x,x)$, the antecedent is true, and we have $\mathfrak{M}, s \vDash X(x)$, which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic second-order sentença of the form φ' is equivalent to a first-order sentença. There is no effective method to decide which ones are.

Capítulo 3

Axiomatic Derivações

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3.1 Introduction

We have a semantics for the basic modal language in terms of modal models, and a notion of a fórmula being valid—true at all worlds in all models—or valid with respect to some class of models or frames—true at all worlds in all models in the class, or based on the frame. Logic usually connects such semantic characterizations of validity with a proof-theoretic notion of derivabilidade. The aim is to define a notion of derivabilidade in some system such that a fórmula is derivável iff it is valid. nml:axs:int:sec

The simplest and historically oldest derivação systems are so-called Hilbert-type or axiomatic derivação systems. Hilbert-type derivação systems for many modal logics are relatively easy to construct: they are simple as objects of metatheoretical study (p.ex., to prove soundness and completeness). However, they are much harder to use to prove fórmulas in than, say, natural deduction systems.

In Hilbert-type derivação systems, a derivação of a fórmula is a sequence of fórmulas leading from certain axioms, via a handful of inference rules, to the fórmula in question. Since we want the derivação system to match the semantics, we have to guarantee that the set of derivável formulas are true in all models (or true in all models in which all axioms are true). We'll first isolate some properties of modal logics that are necessary for this to work: the "normal" modal logics. For normal modal logics, there are only two inference rules that need to be assumed: modus ponens and necessitation. As axioms we take all (substitution instances) of tautologies, and, depending on the modal logic we deal with, a number of modal axioms. Even if we are just interested in the class of all models, we must also count all substitution instances of K and Dual as axioms. This alone generates the minimal normal modal logic **K**.

Definição 3.1. The rule of *modus ponens* is the inference schema

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ \text{MP}$$

We say a fórmula ψ *follows from* fórmulas φ, χ by modus ponens iff $\chi \equiv \varphi \to \psi$.

Definição 3.2. The rule of *necessitation* is the inference schema

$$\frac{\varphi}{\Box\varphi}\ \text{NEC}$$

We say the fórmula ψ follows from the fórmulas φ by necessitation iff $\psi \equiv \Box\varphi$.

Definição 3.3. A *derivação* from a set of axioms Σ is a sequence of fórmulas $\psi_1, \psi_2, \ldots, \psi_n$, where each ψ_i is either

1. a substitution instance of a tautology, or
2. a substitution instance of a fórmula in Σ, or
3. follows from two fórmulas ψ_j, ψ_k with $j, k < i$ by modus ponens, or
4. follows from a fórmula ψ_j with $j < i$ by necessitation.

If there is such a derivação with $\psi_n \equiv \varphi$, we say that φ is *derivável from* Σ, in symbols $\Sigma \vdash \varphi$.

With this definition, it will turn out that the set of derivável formulas forms a normal modal logic, and that any derivável fórmula is true in every model in which every axiom is true. This property of derivações is called *soundness*. The converse, *completeness*, is harder to prove.
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3.2 Normal Modal Logics

nml:prf:nor:sec Not every set of modal fórmulas can easily be characterized as those fórmulas derivable from a set of axioms. We want modal logics to be well-behaved. First of all, everything we can deriva in classical propositional logic should still be derivável, of course taking into account that the fórmulas may now contain also $\Box$ and $\Diamond$. To this end, we require that a modal logic contain all tautological instances and be closed under modus ponens.

Definição 3.4. A *modal logic* is a set Σ of modal fórmulas which

1. contains all tautologies, and
2. is closed under substitution, O.s., if $\varphi \in \Sigma$, and $\theta_1, \ldots, \theta_n$ are fórmulas, then
$$\varphi[\theta_1/p_1, \ldots, \theta_n/p_n] \in \Sigma,$$
3. is closed under *modus ponens*, O.s., if φ and $\varphi \to \psi \in \Sigma$, then $\psi \in \Sigma$.

In order to use the relational semantics for modal logics, we also have to require that all fórmulas valid in all modal models are included. It turns out that this requirement is met as soon as all instances of K and DUAL are derivável, and whenever a fórmula φ is derivável, so is $\Box\varphi$. A modal logic that satisfies these conditions is called *normal*. (Of course, there are also non-normal modal logics, but the usual relational models are not adequate for them.)

Definição 3.5. A modal logic Σ is *normal* if it contains

$$\Box(p \to q) \to (\Box p \to \Box q), \tag{K}$$
$$\Diamond p \leftrightarrow \neg\Box\neg p \tag{DUAL}$$

and is closed under *necessitation*, O.s., if $\varphi \in \Sigma$, then $\Box\varphi \in \Sigma$.

Observe that while tautological implication is "fine-grained" enough to preserve *truth at a world*, the rule NEC only preserves *truth in a model* (and hence also validity in a frame or in a class of frames).

Proposição 3.6. *Every normal modal logic is closed under rule* RK, nml:prf:nor:prop:rk

$$\frac{\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n)\cdots)}{\Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n)\cdots).}\ \text{RK}$$

Prova. By induction on n: If $n = 1$, then the rule is just NEC, and every normal modal logic is closed under NEC.

Now suppose the result holds for $n-1$; we show it holds for n.

Assume

$$\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n)\cdots) \in \Sigma$$

By the induction hypothesis, we have

$$\Box\varphi_1 \to (\Box\varphi_2 \to \cdots \Box(\varphi_{n-1} \to \varphi_n)\cdots) \in \Sigma$$

Since Σ is a normal modal logic, it contains all instances of K, in particular

$$\Box(\varphi_{n-1} \to \varphi_n) \to (\Box\varphi_{n-1} \to \Box\varphi_n) \in \Sigma$$

Using modus ponens and suitable tautological instances we get

$$\Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n)\cdots) \in \Sigma.$$ □

Proposição 3.7. *Every normal modal logic Σ contains $\neg\Diamond\bot$.* nml:prf:nor:prop:notDiamondBot

Problema 3.1. Prove Proposição 3.7.

Proposição 3.8. *Let $\varphi_1, \ldots, \varphi_n$ be fórmulas. Then there is a smallest modal logic Σ containing all instances of $\varphi_1, \ldots, \varphi_n$.*

Prova. Given $\varphi_1, \ldots, \varphi_n$, define Σ as the intersection of all normal modal logics containing all instances of $\varphi_1, \ldots, \varphi_n$. The intersection is non-empty as $\mathrm{Frm}(\mathcal{L})$, the set of all fórmulas, is such a modal logic. □

Definição 3.9. The smallest normal modal logic containing $\varphi_1, \ldots, \varphi_n$ is called a *modal system* and denoted by $\mathbf{K}\varphi_1 \ldots \varphi_n$. The smallest normal modal logic is denoted by $\mathbf{K}$.

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3.3 Derivações and Modal Systems

nml:prf:prf:sec We first define what a derivação is for normal modal logics. Roughly, a derivação is a sequence of fórmulas in which every membro is either (a substitution instance of) one of a number of *axioms*, or follows from previous membros by one of a few inference rules. For normal modal logics, all instances of tautologies, K, and DUAL count as axioms. This results in the modal system $\mathbf{K}$, the smallest normal modal logic. We may wish to add additional axioms to obtain other systems, however. The rules are always modus ponens MP and necessitation NEC.

Definição 3.10. Given a modal system $\mathbf{K}\varphi_1 \ldots \varphi_n$ and a fórmula ψ we say that ψ is *derivável* in $\mathbf{K}\varphi_1 \ldots \varphi_n$, written $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi$, if and only if there are fórmulas $\chi_1, \ldots, \chi_k$ such that $\chi_k = \psi$ and each χ_i is either a tautological instance, or an instance of one of K, DUAL, $\varphi_1, \ldots, \varphi_n$, or it follows from previous fórmulas by means of the rules MP or NEC.

The following proposition allows us to show that $\psi \in \Sigma$ by exhibiting a Σ-derivação of ψ.

Proposição 3.11. $\mathbf{K}\varphi_1 \ldots \varphi_n = \{\psi : \mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi\}$.

Prova. We use induction on the length of derivações to show that $\{\psi : \mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi\} \subseteq \mathbf{K}\varphi_1 \ldots \varphi_n$.

If the derivação of ψ has length 1, it contains a single fórmula. That fórmula cannot follow from previous formulas by MP or NEC, so must be a tautological instance, an instance of K, DUAL, or an instance of one of $\varphi_1, \ldots, \varphi_n$. But $\mathbf{K}\varphi_1 \ldots \varphi_n$ contains these as well, so $\psi \in \mathbf{K}\varphi_1 \ldots \varphi_n$.

If the derivação of ψ has length > 1, then ψ may in addition be obtained by MP or NEC from fórmulas not occurring as the last line in the derivação. If ψ follows from χ and $\chi \to \psi$ (by MP), then χ and $\chi \to \psi \in \mathbf{K}\varphi_1 \ldots \varphi_n$ by induction hypothesis. But every modal logic is closed under modus ponens, so $\psi \in \mathbf{K}\varphi_1 \ldots \varphi_n$. If $\psi \equiv \Box\chi$ follows from χ by NEC, then $\chi \in \mathbf{K}\varphi_1 \ldots \varphi_n$ by induction hypothesis. But every normal modal logic is closed under NEC, so $\psi \in \mathbf{K}\varphi_1 \ldots \varphi_n$.

The converse inclusion follows by showing that $\Sigma = \{\psi : \mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi\}$ is a normal modal logic containing all the instances of $\varphi_1, \ldots, \varphi_n$, and the observation that $\mathbf{K}\varphi_1 \ldots \varphi_n$ is, by definition, the smallest such logic.

1. Every tautology ψ is a tautological instance, so $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi$, so Σ contains all tautologies.

2. If $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \chi$ and $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \chi \to \psi$, then $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi$: Combine the derivação of χ with that of $\chi \to \psi$, and add the line ψ. The last line is justified by MP. So Σ is closed under modus ponens.

3. If ψ has a derivação, then every substitution instance of ψ also has a derivação: apply the substitution to every fórmula in the derivação. (Exercise: prove by induction on the length of derivações that the result is also a correct derivação). So Σ is closed under uniform substitution. (We have now established that Σ satisfies all conditions of a modal logic.)

4. We have $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash$ K, so $K \in \Sigma$.

5. We have $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash$ DUAL, so DUAL $\in \Sigma$.

6. If $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \chi$, the additional line $\Box\chi$ is justified by NEC. Consequently, Σ is closed under NEC. Thus, Σ is normal. □

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3.4 Proofs in K

In order to practice proofs in the smallest modal system, we show the valid fórmulas on the left-hand side of Tabela 1.1 can all be given **K**-proofs. nml:prf:prk:sec

Proposição 3.12. $\mathbf{K} \vdash \Box\varphi \to \Box(\psi \to \varphi)$

Prova.

1.	$\varphi \to (\psi \to \varphi)$	TAUT
2.	$\Box(\varphi \to (\psi \to \varphi))$	NEC, 1
3.	$\Box(\varphi \to (\psi \to \varphi)) \to (\Box\varphi \to \Box(\psi \to \varphi))$	K
4.	$\Box\varphi \to \Box(\psi \to \varphi)$	MP, 2, 3

□

Proposição 3.13. $\mathbf{K} \vdash \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$

Prova.

$$
\begin{array}{lll}
1. & (\varphi \wedge \psi) \to \varphi & \textsc{taut} \\
2. & \Box((\varphi \wedge \psi) \to \varphi) & \textsc{nec} \\
3. & \Box((\varphi \wedge \psi) \to \varphi) \to (\Box(\varphi \wedge \psi) \to \Box\varphi) & \mathrm{K} \\
4. & \Box(\varphi \wedge \psi) \to \Box\varphi & \textsc{mp}, 2, 3 \\
5. & (\varphi \wedge \psi) \to \psi & \textsc{taut} \\
6. & \Box((\varphi \wedge \psi) \to \psi) & \textsc{nec} \\
7. & \Box((\varphi \wedge \psi) \to \psi) \to (\Box(\varphi \wedge \psi) \to \Box\psi) & \mathrm{K} \\
8. & \Box(\varphi \wedge \psi) \to \Box\psi & \textsc{mp}, 6, 7 \\
9. & (\Box(\varphi \wedge \psi) \to \Box\varphi) \to & \\
 & \quad ((\Box(\varphi \wedge \psi) \to \Box\psi) \to & \\
 & \quad (\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi))) & \textsc{taut} \\
10. & (\Box(\varphi \wedge \psi) \to \Box\psi) \to & \\
 & \quad (\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)) & \textsc{mp}, 4, 9 \\
11. & \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi) & \textsc{mp}, 8, 10.
\end{array}
$$

Note that the fórmula on line 9 is an instance of the tautology

$$(p \to q) \to ((p \to r) \to (p \to (q \wedge r))).$$ □

Proposição 3.14. $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

Prova.

$$
\begin{array}{lll}
1. & \varphi \to (\psi \to (\varphi \wedge \psi)) & \textsc{taut} \\
2. & \Box(\varphi \to (\psi \to (\varphi \wedge \psi))) & \textsc{nec}, 1 \\
3. & \Box(\varphi \to (\psi \to (\varphi \wedge \psi))) \to (\Box\varphi \to \Box(\psi \to (\varphi \wedge \psi))) & \mathrm{K} \\
4. & \Box\varphi \to \Box(\psi \to (\varphi \wedge \psi)) & \textsc{mp}, 2, 3 \\
5. & \Box(\psi \to (\varphi \wedge \psi)) \to (\Box\psi \to \Box(\varphi \wedge \psi)) & \mathrm{K} \\
6. & (\Box\varphi \to \Box(\psi \to (\varphi \wedge \psi))) \to & \\
 & \quad (\Box(\psi \to (\varphi \wedge \psi)) \to (\Box\psi \to \Box(\varphi \wedge \psi))) \to & \\
 & \quad (\Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi)))) & \textsc{taut} \\
7. & (\Box(\psi \to (\varphi \wedge \psi)) \to (\Box\psi \to \Box(\varphi \wedge \psi))) \to & \\
 & \quad (\Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi))) & \textsc{mp}, 4, 6 \\
8. & \Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi))) & \textsc{mp}, 5, 7 \\
9. & (\Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi)))) \to & \\
 & \quad ((\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)) & \textsc{taut} \\
10. & (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi) & \textsc{mp}, 8, 9
\end{array}
$$

The fórmulas on lines 6 and 9 are instances of the tautologies

$$(p \to q) \to ((q \to r) \to (p \to r))$$
$$(p \to (q \to r)) \to ((p \wedge q) \to r)$$ □

Proposição 3.15. $\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$

Prova.

1.	$\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p$	DUAL
2.	$(\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p) \to$ $(\neg\Box\neg\neg p \to \Diamond\neg p)$	TAUT
3.	$\neg\Box\neg\neg p \to \Diamond\neg p$	MP, 1, 2
4.	$\neg\neg p \to p$	TAUT
5.	$\Box(\neg\neg p \to p)$	NEC, 4
6.	$\Box(\neg\neg p \to p) \to (\Box\neg\neg p \to \Box p)$	K
7.	$(\Box\neg\neg p \to \Box p)$	MP, 5, 6
8.	$(\Box\neg\neg p \to \Box p) \to (\neg\Box p \to \neg\Box\neg\neg p)$	TAUT
9.	$\neg\Box p \to \neg\Box\neg\neg p$	MP, 7, 8
10.	$(\neg\Box p \to \neg\Box\neg\neg p) \to$ $((\neg\Box\neg\neg p \to \Diamond\neg p) \to (\neg\Box p \to \Diamond\neg p))$	TAUT
11.	$(\neg\Box\neg\neg p \to \Diamond\neg p) \to (\neg\Box p \to \Diamond\neg p)$	MP, 9, 10
12.	$\neg\Box p \to \Diamond\neg p$	MP, 3, 11

The fórmulas on lines 8 and 10 are instances of the tautologies

$$(p \to q) \to (\neg q \to \neg p)$$
$$(p \to q) \to ((q \to r) \to (p \to r)). \qquad \Box$$

Problema 3.2. Find derivações in **K** for the following fórmulas:

1. $\Box\neg p \to \Box(p \to q)$
2. $(\Box p \lor \Box q) \to \Box(p \lor q)$
3. $\Diamond p \to \Diamond(p \lor q)$

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3.5 Derived Rules

Finding and writing derivações is obviously difficult, cumbersome, and repetitive. For instance, very often we want to pass from $\varphi \to \psi$ to $\Box\varphi \to \Box\psi$, O.s., apply rule RK. That requires an application of NEC, then recording the proper instance of K, then applying MP. Passing from $\varphi \to \psi$ and $\psi \to \chi$ to $\varphi \to \chi$ requires recording the (long) tautological instance nml:prf:der:sec

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$

and applying MP twice. Often we want to replace a sub-fórmula by a formula we know to be equivalent, p.ex., $\Diamond\varphi$ by $\neg\Box\neg\varphi$, or $\neg\neg\varphi$ by φ. So rather than write out the actual derivação, it is more convenient to simply record why the intermediate steps are derivável. For this purpose, let us collect some facts about derivabilidade.

Proposição 3.16. *If* $\mathbf{K} \vdash \varphi_1, \ldots, \mathbf{K} \vdash \varphi_n$*, and* ψ *follows from* $\varphi_1, \ldots, \varphi_n$ *by propositional logic, then* $\mathbf{K} \vdash \psi$*.*

Prova. If ψ follows from $\varphi_1, \ldots, \varphi_n$ by propositional logic, then

$$\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \psi) \ldots)$$

is a tautological instance. Applying MP n times gives a derivação of ψ. $\square$

We will indicate use of this proposition by PL.

Proposição 3.17. *If* $\mathbf{K} \vdash \varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \varphi_n) \ldots)$ *then* $\mathbf{K} \vdash \Box\varphi_1 \to (\Box\varphi_2 \to \cdots (\Box\varphi_{n-1} \to \Box\varphi_n) \ldots)$.

Prova. By induction on n, just as in the proof of Proposição 3.6. $\square$

We will indicate use of this proposition by RK. Let's illustrate how these results help establishing derivabilidade results more easily.

Proposição 3.18. $\mathbf{K} \vdash (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$

Prova.

1.	$\mathbf{K} \vdash \varphi \to (\psi \to (\varphi \land \psi))$	TAUT
2.	$\mathbf{K} \vdash \Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi)))$	RK, 1
3.	$\mathbf{K} \vdash (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$	PL, 2

$\square$

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Proposição 3.19. *If* $\mathbf{K} \vdash \varphi \leftrightarrow \psi$ *and* $\mathbf{K} \vdash \chi[\varphi/q]$ *then* $\mathbf{K} \vdash \chi[B/q]$

Prova. Exercise. $\square$

Problema 3.3. Prove Proposição 3.19 by proving, by induction on the complexity of χ, that if $\mathbf{K} \vdash \varphi \leftrightarrow \psi$ then $\mathbf{K} \vdash \chi[\varphi/q] \leftrightarrow \chi[\psi/q]$.

This proposition comes in handy especially when we want to convert $\Diamond$ into $\Box$ (or vice versa), or remove double negations inside a fórmula. In what follows, we will mark applications of Proposição 3.19 by "φ for ψ" whenever we re-write a fórmula $\chi(\psi)$ for $\chi(\varphi)$. In other words, "φ for ψ" abbreviates:

$\vdash \chi(\varphi)$	
$\vdash \varphi \leftrightarrow \psi$	
$\vdash \chi(\psi)$	by Proposição 3.19

For instance:

Proposição 3.20. $\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$

Prova.

1.	$\mathbf{K} \vdash \Diamond\neg p \leftrightarrow \neg\Box\neg\neg p$	DUAL
2.	$\mathbf{K} \vdash \neg\Box\neg\neg p \to \Diamond\neg p$	PL, 1
3.	$\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$	p for $\neg\neg p$

$\square$

In the above derivação, the final step "p for $\neg\neg p$" is short for

$$\begin{array}{ll} \mathbf{K} \vdash \neg\Box\neg\neg p \to \Diamond\neg p & \\ \mathbf{K} \vdash \neg\neg p \leftrightarrow p & \textsc{taut} \\ \mathbf{K} \vdash \neg\Box p \to \Diamond\neg p & \text{by Proposição 3.19} \end{array}$$

The roles of $\chi(q)$, φ, and ψ in Proposição 3.19 are played here, respectively, by $\neg\Box q \to \Diamond\neg p$, $\neg\neg p$, and p.

When a fórmula contains a sub-fórmula $\neg\Diamond\varphi$, we can replace it by $\Box\neg\varphi$ using Proposição 3.19, since $\mathbf{K} \vdash \neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$. We'll indicate this and similar replacements simply by "$\Box\neg$ for $\neg\Diamond$."

The following proposition justifies that we can establish derivabilidade results schematically. P.ex., the previous proposition does not just establish that $\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$, but $\mathbf{K} \vdash \neg\Box\varphi \to \Diamond\neg\varphi$ for arbitrary φ.

Proposição 3.21. *If φ is a substitution instance of ψ and $\mathbf{K} \vdash \psi$, then $\mathbf{K} \vdash \varphi$.*

Prova. It is tedious but routine to verify (by induction on the length of the derivação of ψ) that applying a substitution to an entire derivação also results in a correct derivação. Specifically, substitution instances of tautological instances are themselves tautological instances, substitution instances of instances of DUAL and K are themselves instances of DUAL and K, and applications of MP and NEC remain correct when substituting fórmulas for variáveis proposicionais in both premise(s) and conclusion. □

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3.6 More Proofs in K

Let's see some more examples of derivabilidade in $\mathbf{K}$, now using the simplified method introduced in Seção 3.5. nml:prf:mpr:sec

Proposição 3.22. $\mathbf{K} \vdash \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$

Prova.

$$\begin{array}{lll} 1. & \mathbf{K} \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi) & \textsc{pl} \\ 2. & \mathbf{K} \vdash \Box(\varphi \to \psi) \to (\Box\neg\psi \to \Box\neg\varphi) & \textsc{rk}, 1 \\ 3. & \mathbf{K} \vdash (\Box\neg\psi \to \Box\neg\varphi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi) & \textsc{taut} \\ 4. & \mathbf{K} \vdash \Box(\varphi \to \psi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi) & \textsc{pl}, 2, 3 \\ 5. & \mathbf{K} \vdash \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi) & \Diamond \text{ for } \neg\Box\neg. \end{array}$$

□

Proposição 3.23. $\mathbf{K} \vdash \Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$

Prova.

1.	$\mathbf{K} \vdash \varphi \to (\neg\psi \to \neg(\varphi \to \psi))$	TAUT
2.	$\mathbf{K} \vdash \Box\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \to \psi))$	RK, 1
3.	$\mathbf{K} \vdash \Box\varphi \to (\neg\Box\neg(\varphi \to \psi) \to \neg\Box\neg\psi)$	PL, 2
4.	$\mathbf{K} \vdash \Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$	$\Diamond$ for $\neg\Box\neg$.

□

Proposição 3.24. $\mathbf{K} \vdash (\Diamond\varphi \lor \Diamond\psi) \to \Diamond(\varphi \lor \psi)$

Prova.

1.	$\mathbf{K} \vdash \neg(\varphi \lor \psi) \to \neg\varphi$	TAUT
2.	$\mathbf{K} \vdash \Box\neg(\varphi \lor \psi) \to \Box\neg\varphi$	RK, 1
3.	$\mathbf{K} \vdash \neg\Box\neg\varphi \to \neg\Box\neg(\varphi \lor \psi)$	PL, 2
4.	$\mathbf{K} \vdash \Diamond\varphi \to \Diamond(\varphi \lor \psi)$	$\Diamond$ for $\neg\Box\neg$
5.	$\mathbf{K} \vdash \Diamond\psi \to \Diamond(\varphi \lor \psi)$	similarly
6.	$\mathbf{K} \vdash (\Diamond\varphi \lor \Diamond\psi) \to \Diamond(\varphi \lor \psi)$	PL, 4, 5.

□

Proposição 3.25. $\mathbf{K} \vdash \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi)$

Prova.

1.	$\mathbf{K} \vdash \neg\varphi \to (\neg\psi \to \neg(\varphi \lor \psi))$	TAUT
2.	$\mathbf{K} \vdash \Box\neg\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \lor \psi))$	RK
3.	$\mathbf{K} \vdash \Box\neg\varphi \to (\neg\Box\neg(\varphi \lor \psi) \to \neg\Box\neg\psi)$	PL, 2
4.	$\mathbf{K} \vdash \neg\Box\neg(\varphi \lor \psi) \to (\Box\neg\varphi \to \neg\Box\neg\psi)$	PL, 3
5.	$\mathbf{K} \vdash \neg\Box\neg(\varphi \lor \psi) \to (\neg\neg\Box\neg\psi \to \neg\Box\neg\varphi)$	PL, 4
6.	$\mathbf{K} \vdash \Diamond(\varphi \lor \psi) \to (\neg\Diamond\psi \to \Diamond\varphi)$	$\Diamond$ for $\neg\Box\neg$
7.	$\mathbf{K} \vdash \Diamond(\varphi \lor \psi) \to (\Diamond\psi \lor \Diamond\varphi)$	PL, 6.

□

Problema 3.4. Show that the following derivabilidade claims hold:

1. $\mathbf{K} \vdash \Diamond\neg\bot \to (\Box\varphi \to \Diamond\varphi)$;
2. $\mathbf{K} \vdash \Box(\varphi \lor \psi) \to (\Diamond\varphi \lor \Box\psi)$;
3. $\mathbf{K} \vdash (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$.

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3.7 Dual Fórmulas

nml:prf:dua:sec nml:prf:dua:def:duals

Definição 3.26. Each of the fórmulas T, B, 4, and 5 has a *dual*, denoted by a subscripted diamond, as follows:

$$p \to \Diamond p \qquad (\mathrm{T}_\Diamond)$$

$$\Diamond\Box p \to p \qquad (\mathrm{B}_\Diamond)$$

$$\Diamond\Diamond p \to \Diamond p \qquad (4_\Diamond)$$

$$\Diamond\Box p \to \Box p \qquad (5_\Diamond)$$

Each of the above dual fórmulas is obtained from the corresponding fórmula by substituting $\neg p$ for p, contraposing, replacing $\neg\Box\neg$ by $\Diamond$, and replacing $\neg\Diamond\neg$ by $\Box$. D, O.s., $\Box\varphi \to \Diamond\varphi$ is its own dual in that sense.

Proposição 3.27. *For each fórmula φ in Definição 3.26: $\mathbf{K}\varphi = \mathbf{K}\varphi_\Diamond$.* nml:prf:dua: prop:dualsys

Prova. Exercise. □

Problema 3.5. Prove Proposição 3.27.

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3.8 Proofs in Modal Systems

We now come to proofs in systems of modal logic other than **K**. nml:prf:prs: sec

Proposição 3.28. *The following provability results obtain:* nml:prf:prs: prop:S5facts

1. $\mathbf{KT5} \vdash$ B;
2. $\mathbf{KT5} \vdash 4$;
3. $\mathbf{KDB4} \vdash$ T;
4. $\mathbf{KB4} \vdash 5$;
5. $\mathbf{KB5} \vdash 4$;
6. $\mathbf{KT} \vdash$ D. nml:prf:prs: prop:S5facts-KT-D

Prova. We exhibit proofs for each.

1. $\mathbf{KT5} \vdash$ B:

1.	$\mathbf{KT5} \vdash \Diamond\varphi \to \Box\Diamond\varphi$	5
2.	$\mathbf{KT5} \vdash \varphi \to \Diamond\varphi$	$\mathrm{T}_\Diamond$
3.	$\mathbf{KT5} \vdash \varphi \to \Box\Diamond\varphi$	PL, 2, 1.

2. $\mathbf{KT5} \vdash 4$:

1.	$\mathbf{KT5} \vdash \Diamond\Box\varphi \to \Box\Diamond\Box\varphi$	5 with $\Box\varphi$ for p
2.	$\mathbf{KT5} \vdash \Box\varphi \to \Diamond\Box\varphi$	$\mathrm{T}_\Diamond$ with $\Box\varphi$ for p
3.	$\mathbf{KT5} \vdash \Box\varphi \to \Box\Diamond\Box\varphi$	PL, 2, 1
4.	$\mathbf{KT5} \vdash \Diamond\Box\varphi \to \Box\varphi$	$5_\Diamond$
5.	$\mathbf{KT5} \vdash \Box\Diamond\Box\varphi \to \Box\Box\varphi$	RK, 4
6.	$\mathbf{KT5} \vdash \Box\varphi \to \Box\Box\varphi$	PL, 3, 5.

3. $\mathbf{KDB4} \vdash$ T:

1. $\mathbf{KDB4} \vdash \Diamond\Box\varphi \to \varphi$ $\mathrm{B}_\Diamond$
2. $\mathbf{KDB4} \vdash \Box\Box\varphi \to \Diamond\Box\varphi$ D with $\Box\varphi$ for p
3. $\mathbf{KDB4} \vdash \Box\Box\varphi \to \varphi$ PL, 1, 2
4. $\mathbf{KDB4} \vdash \Box\varphi \to \Box\Box\varphi$ 4
5. $\mathbf{KDB4} \vdash \Box\varphi \to \varphi$ PL, 4, 3.

4. $\mathbf{KB4} \vdash 5$:

1. $\mathbf{KB4} \vdash \Diamond\varphi \to \Box\Diamond\Diamond\varphi$ B with $\Diamond\varphi$ for p
2. $\mathbf{KB4} \vdash \Diamond\Diamond\varphi \to \Diamond\varphi$ $4_\Diamond$
3. $\mathbf{KB4} \vdash \Box\Diamond\Diamond\varphi \to \Box\Diamond\varphi$ RK, 2
4. $\mathbf{KB4} \vdash \Diamond\varphi \to \Box\Diamond\varphi$ PL, 1, 3.

5. $\mathbf{KB5} \vdash 4$:

1. $\mathbf{KB5} \vdash \Box\varphi \to \Box\Diamond\Box\varphi$ B with $\Box\varphi$ for p
2. $\mathbf{KB5} \vdash \Diamond\Box\varphi \to \Box\varphi$ $5_\Diamond$
3. $\mathbf{KB5} \vdash \Box\Diamond\Box\varphi \to \Box\Box\varphi$ RK, 2
4. $\mathbf{KB5} \vdash \Box\varphi \to \Box\Box\varphi$ PL, 1, 3.

6. $\mathbf{KT} \vdash \mathrm{D}$:

1. $\mathbf{KT} \vdash \Box\varphi \to \varphi$ T
2. $\mathbf{KT} \vdash \varphi \to \Diamond\varphi$ $\mathrm{T}_\Diamond$
3. $\mathbf{KT} \vdash \Box\varphi \to \Diamond\varphi$ PL, 1, 2 □

Definição 3.29. Following tradition, we define **S4** to be the system **KT4**, and **S5** the system **KTB4**.

The following proposition shows that the classical system **S5** has several equivalent axiomatizations. This should not surprise, as the various combinations of axioms all characterize equivalence relations (see Proposição 2.12).

nml:prf:prs: prop:S5

Proposição 3.30. **KTB4 = KT5 = KDB4 = KDB5**.

Prova. Exercise. □

Problema 3.6. Prove Proposição 3.30.

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3.9 Soundness

A derivação system is called sound if everything that can be derivad is valid. When considering modal systems, O.s., derivações where in addition to K we can use instances of some fórmulas $\varphi_1, \ldots, \varphi_n$, we want every derivável formula to be true in any model in which $\varphi_1, \ldots, \varphi_n$ are true. nml:prf:snd:sec

Teorema 3.31 (Soundness Theorem). *If every instance of $\varphi_1, \ldots, \varphi_n$ is valid in the classes of models $\mathcal{C}_1, \ldots, \mathcal{C}_n$, respectively, then $\mathbf{K}\varphi_1 \ldots \varphi_n \vdash \psi$ implies that ψ is valid in the class of models $\mathcal{C}_1 \cap \cdots \cap \mathcal{C}_n$.* nml:prf:snd:thm:soundness

Prova. By induction on length of proofs. For brevity, put $\mathcal{C} = \mathcal{C}_1 \cap \cdots \cap \mathcal{C}_n$.

1. Induction Basis: If ψ has a proof of length 1, then it is either a tautological instance, an instance of K, or of DUAL, or an instance of one of $\varphi_1, \ldots, \varphi_n$. In the first case, ψ is valid in $\mathcal{C}$, since tautological instance are valid in *any* class of models, by Proposição 1.15. Similarly in the second case, by Proposição 1.18 and Proposição 1.19. Finally in the third case, since ψ is valid in $\mathcal{C}_i$ and $\mathcal{C} \subseteq \mathcal{C}_i$, we have that ψ is valid in $\mathcal{C}$ as well by Proposição 1.11.

2. Inductive step: Suppose ψ has a proof of length $k > 1$. If ψ is a tautological instance or an instance of one of $\varphi_1, \ldots, \varphi_n$, we proceed as in the previous step. So suppose ψ is obtained by MP from previous fórmulas $\chi \to \psi$ and χ. Then $\chi \to \psi$ and χ have proofs of length $< k$, and by inductive hypothesis they are valid in $\mathcal{C}$. By Proposição 1.20, ψ is valid in $\mathcal{C}$ as well. Finally suppose ψ is obtained by NEC from χ (so that $\psi = \Box\chi$). By inductive hypothesis, χ is valid in $\mathcal{C}$, and by Proposição 1.12 so is ψ.
□

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3.10 Showing Systems are Distinct

In Seção 3.8 we saw how to prove that two systems of modal logic are in fact the same system. Teorema 3.31 allows us to show that two modal systems Σ and Σ' are distinct, by finding a fórmula φ such that $\Sigma' \vdash \varphi$ that fails in a model of Σ. nml:prf:dis:sec

Proposição 3.32. $\mathbf{KD} \subsetneq \mathbf{KT}$

Prova. This is the syntactic counterpart to the semantic fact that all reflexive relations are serial. To show $\mathbf{KD} \subseteq \mathbf{KT}$ we need to see that $\mathbf{KD} \vdash \psi$ implies $\mathbf{KT} \vdash \psi$, which follows from $\mathbf{KT} \vdash \mathbf{D}$, as shown in Proposição 3.28(6). To show that the inclusion is proper, by Soundness (Teorema 3.31), it suffices to exhibit a model of **KD** where T, O.s., $\Box p \to p$, fails (an easy task left as an exercise), for then by Soundness $\mathbf{KD} \nvdash \Box p \to p$. □

Proposição 3.33. $\mathbf{KB} \neq \mathbf{K4}$.

Prova. We construct a symmetric model where some instance of 4 fails; since obviously the instance is derivável for **K4** but not in **KB**, it will follow $\mathbf{K4} \not\subseteq \mathbf{KB}$. Consider the symmetric model $\mathfrak{M}$ of Figura 3.1. Since the model is symmetric, K and B are true in $\mathfrak{M}$ (by Proposição 1.18 and Teorema 2.1, respectively). However, $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$. $\Box$

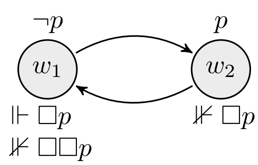

Figura 3.1: A symmetric model falsifying an instance of 4.

nml:prf:dis: fig:Bnot4 nml:prf:dis: thm:KTBnot45

Teorema 3.34. $\mathbf{KTB} \nvdash \mathbf{4}$ *and* $\mathbf{KTB} \nvdash \mathbf{5}$.

Prova. By Teorema 2.1 we know that all instances of T and B are true in every reflexive symmetric model (respectively). So by soundness, it suffices to find a reflexive symmetric model containing a world at which some instance of 4 fails, and similarly for 5. We use the same model for both claims. Consider the symmetric, reflexive model in Figura 3.2. Then $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$, so 4 fails at w_1. Similarly, $\mathfrak{M}, w_2 \nVdash \Diamond\neg p \to \Box\Diamond\neg p$, so the instance of 5 with $\varphi = \neg p$ fails at w_2. $\Box$

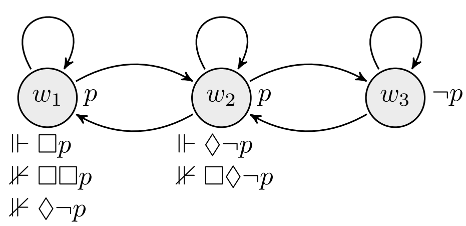

Figura 3.2: The model for Teorema 3.34.

nml:prf:dis: fig:KTBnot45 nml:prf:dis: thm:KD5not4

Teorema 3.35. $\mathbf{KD5} \neq \mathbf{KT4} = \mathbf{S4}$.

Prova. By Teorema 2.1 we know that all instances of D and 5 are true in all serial euclidean models. So it suffices to find a serial euclidean model containing a world at which some instance of 4 fails. Consider the model of Figura 3.3, and notice that $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$. $\Box$

Problema 3.7. Give an alternative proof of Teorema 3.35 using a model with 3 worlds.

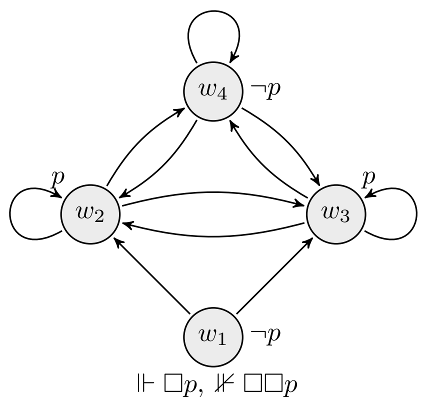

Figura 3.3: The model for Teorema 3.35. nml:prf:dis:fig:KD5not4

Problema 3.8. Provide a single reflexive transitive model showing that both **KT4** $\nvdash$ B and **KT4** $\nvdash$ 5.

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3.11 Derivabilidade from a Set of Fórmulas

In Seção 3.8 we defined a notion of provability of a fórmula in a system Σ. We now extend this notion to provability in Σ from fórmulas in a set Γ. nml:prf:prg:sec

Definição 3.36. A fórmula φ is derivável in a system Σ from a set of fórmulas Γ, written $\Gamma \vdash_\Sigma \varphi$ if and only if there are $\psi_1, \dots, \psi_n \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$. nml:prf:prg:defn:Gammaproves

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3.12 Properties of Derivabilidade

nml:prf:prp:sec

Proposição 3.37. *Let Σ be a modal system and Γ a set of modal fórmulas. The following properties hold:* *nml:prf:prp:prop:derivabilityfacts*

1. Monotonicity*: If $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_\Sigma \varphi$;* *nml:prf:prp:prop:derivabilityfacts-monotonicity*
2. Reflexivity*: If $\varphi \in \Gamma$ then $\Gamma \vdash_\Sigma \varphi$;* *nml:prf:prp:prop:derivabilityfacts-reflexivity*
3. Cut*: If $\Gamma \vdash_\Sigma \varphi$ and $\Delta \cup \{\varphi\} \vdash_\Sigma \psi$ then $\Gamma \cup \Delta \vdash_\Sigma \psi$;* *nml:prf:prp:prop:derivabilityfacts-cut*
4. Deduction theorem*: $\Gamma \cup \{\psi\} \vdash_\Sigma \varphi$ if and only if $\Gamma \vdash_\Sigma \psi \to \varphi$;* *nml:prf:prp:prop:derivabilityfacts-deduction*
5. *$\Gamma \vdash_\Sigma \varphi_1$ and ... and $\Gamma \vdash_\Sigma \varphi_n$ and $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \psi) \cdots)$ is a tautological instance, then $\Gamma \vdash_\Sigma \psi$.* *nml:prf:prp:prop:derivabilityfacts-ruleT*

The proof is an easy exercise. Part (5) of Proposição 3.37 gives us that, for instance, if $\Gamma \vdash_{\Sigma} \varphi \lor \psi$ and $\Gamma \vdash_{\Sigma} \neg\varphi$, then $\Gamma \vdash_{\Sigma} \psi$. Also, in what follows, we write $\Gamma, \varphi \vdash_{\Sigma} \psi$ instead of $\Gamma \cup \{\varphi\} \vdash_{\Sigma} \psi$.

Definição 3.38. A set Γ is *deductively closed* relatively to a system Σ if and only if $\Gamma \vdash_{\Sigma} \varphi$ implies $\varphi \in \Gamma$.

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3.13 Consistency

nml:prf:con: sec Consistency is an important property of sets of fórmulas. A set of fórmulas is inconsistent if a contradiction, such as $\bot$, is derivável from it; and otherwise consistent. If a set is inconsistent, its fórmulas cannot all be true in a model at a world. For the completeness theorem we prove the converse: every consistent set is true at a world in a model, namely in the "canonical model."

Definição 3.39. A set Γ is *consistent* relatively to a system Σ or, as we will say, Σ-consistent, if and only if $\Gamma \nvdash_{\Sigma} \bot$.

So for instance, the set $\{\Box(p \to q), \Box p, \neg\Box q\}$ is consistent relatively to propositional logic, but not **K**-consistent. Similarly, the set $\{\Diamond p, \Box\Diamond p \to q, \neg q\}$ is not **K5**-consistent.

nml:prf:con: prop:consistencyfacts **Proposição 3.40.** *Let Γ be a set of fórmulas. Then:*

1. *Γ is Σ-consistent if and only if there is some fórmula φ such that $\Gamma \nvdash_{\Sigma} \varphi$.*
2. *nml:prf:con: prop:consistencyfacts-b* *$\Gamma \vdash_{\Sigma} \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is not Σ-consistent.*
3. *nml:prf:con: prop:consistencyfacts-c* *If Γ is Σ-consistent, then for any fórmula φ, either $\Gamma \cup \{\varphi\}$ is Σ-consistent or $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent.*

Prova. These facts follow easily using classical propositional logic. We give the argument for (3). Proceed contrapositively and suppose neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent. Then by (2), both $\Gamma, \varphi \vdash_{\Sigma} \bot$ and $\Gamma, \neg\varphi \vdash_{\Sigma} \bot$. By the deduction theorem $\Gamma \vdash_{\Sigma} \varphi \to \bot$ and $\Gamma \vdash_{\Sigma} \neg\varphi \to \bot$. But $(\varphi \to \bot) \to ((\neg\varphi \to \bot) \to \bot)$ is a tautological instance, hence by Proposição 3.37(5), $\Gamma \vdash_{\Sigma} \bot$. □

Capítulo 4

Completeness and Canonical Models

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4.1 Introduction

If Σ is a modal system, then the soundness theorem establishes that if $\Sigma \vdash \varphi$, then φ is valid in any class $\mathcal{C}$ of models in which all instances of all fórmulas in Σ are valid. In particular that means that if $\mathbf{K} \vdash \varphi$ then φ is true in all models; if $\mathbf{KT} \vdash \varphi$ then φ is true in all reflexive models; if $\mathbf{KD} \vdash \varphi$ then φ is true in all serial models, etc. nml:com:int:sec

Completeness is the converse of soundness: that $\mathbf{K}$ is complete means that if a fórmula φ is valid, $\vdash \varphi$, for instance. Proving completeness is a lot harder to do than proving soundness. It is useful, first, to consider the contrapositive: $\mathbf{K}$ is complete iff whenever $\nvdash \varphi$, there is a countermodel, O.s., a model $\mathfrak{M}$ such that $\mathfrak{M} \nVdash \varphi$. Equivalently (negating φ), we could prove that whenever $\nvdash \neg\varphi$, there is a model of φ. In the construction of such a model, we can use information contained in φ. When we find models for specific fórmulas we often do the same: p.ex., if we want to find a countermodel to $p \to \Box q$, we know that it has to contain a world where p is true and $\Box q$ is false. And a world where $\Box q$ is false means there has to be a world accessible from it where q is false. And that's all we need to know: which worlds make the variáveis proposicionais true, and which worlds are accessible from which worlds.

In the case of proving completeness, however, we don't have a specific fórmula φ for which we are constructing a model. We want to establish that a model exists for every φ such that $\nvdash_\Sigma \neg\varphi$. This is a minimal requirement, since *if* $\vdash_\Sigma \neg\varphi$, by soundness, there is no model for φ (in which Σ is true). Now note that $\nvdash_\Sigma \neg\varphi$ iff φ is Σ-consistent. (Recall that $\Sigma \nvdash_\Sigma \neg\varphi$ and $\varphi \nvdash_\Sigma \bot$ are equivalent.) So our task is to construct a model for every Σ-consistent fórmula.

The trick we'll use is to find a Σ-consistent set of fórmulas that contains φ, but also other formulas which tell us what the world that makes φ true has to look like. Such sets are *complete* Σ-consistent sets. It's not enough to construct a model with a single world to make φ true, it will have to contain multiple worlds and an accessibility relation. The complete Σ-consistent set containing φ will also contain other fórmulas of the form $\Box\psi$ and $\Diamond\chi$. In all accessible worlds, ψ has to be true; in at least one, χ has to be true. In order to accomplish this, we'll simply take *all* possible complete Σ-consistent sets as the basis for the set of worlds. A tricky part will be to figure out when a complete Σ-consistent set should count as being accessible from another in our model.

We'll show that in the model so defined, φ is true at a world—which is also a complete Σ-consistent set—iff φ is um membro of that set. If φ is Σ-consistent, it will be um membro of at least one complete Σ-consistent set (a fact we'll prove), and so there will be a world where φ is true. So we will have a single model where every Σ-consistent fórmula φ is true at some world. This single model is the *canonical* model for Σ.

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4.2 Complete Σ-Consistent Sets

nml:com:ccs:sec

Suppose Σ is a set of modal fórmulas—think of them as the axioms or defining principles of a normal modal logic. A set Γ is Σ-consistent iff $\Gamma \nvdash_{\Sigma} \bot$, O.s., if there is no derivação of $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_n \to \bot) \dots)$ from Σ, where each $\varphi_i \in \Gamma$. We will construct a "canonical" model in which each world is taken to be a special kind of Σ-consistent set: one which is not just Σ-consistent, but maximally so, in the sense that it settles the truth value of every modal fórmula: for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$:

Definição 4.1. A set Γ is *complete Σ-consistent* if and only if it is Σ-consistent and for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Complete Σ-consistent sets Γ have a number of useful properties. For one, they are deductively closed, O.s., if $\Gamma \vdash_{\Sigma} \varphi$ then $\varphi \in \Gamma$. This means in particular that every instance of a fórmula $\varphi \in \Sigma$ is also $\in \Gamma$. Moreover, membership in Γ mirrors the truth conditions for the propositional connectives. This will be important when we define the "canonical model."

nml:com:ccs:prop:ccs-properties

Proposição 4.2. *Suppose Γ is complete Σ-consistent. Then:*

nml:com:ccs:prop:ccs-closed 1. *Γ is deductively closed in Σ.*

nml:com:ccs:prop:ccs-sigma 2. *$\Sigma \subseteq \Gamma$.*

nml:com:ccs:prop:ccs-lfalse 3. *$\bot \notin \Gamma$*

nml:com:ccs:prop:ccs-ltrue 4. *$\top \in \Gamma$*

5. *$\lnot\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.* nml:com:ccs: prop:ccs-lnot

6. *$\varphi \land \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$* nml:com:ccs: prop:ccs-land

7. *$\varphi \lor \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$* nml:com:ccs: prop:ccs-lor

8. *$\varphi \to \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$* nml:com:ccs: prop:ccs-lif

9. *$\varphi \leftrightarrow \psi \in \Gamma$ iff either $\varphi \in \Gamma$ and $\psi \in \Gamma$, or $\varphi \notin \Gamma$ and $\psi \notin \Gamma$* nml:com:ccs: prop:ccs-liff

Prova.

1. Suppose $\Gamma \vdash_\Sigma \varphi$ but $\varphi \notin \Gamma$. Then since Γ is complete Σ-consistent, $\lnot\varphi \in \Gamma$. This would make Γ inconsistent, since $\varphi, \lnot\varphi \vdash_\Sigma \bot$.

2. If $\varphi \in \Sigma$ then $\Gamma \vdash_\Sigma \varphi$, and $\varphi \in \Gamma$ by deductive closure, O.s., case (1).

3. If $\bot \in \Gamma$, then $\Gamma \vdash_\Sigma \bot$, so Γ would be Σ-inconsistent.

4. $\Gamma \vdash_\Sigma \top$, so $\top \in \Gamma$ by deductive closure, O.s., case (1).

5. If $\lnot\varphi \in \Gamma$, then by consistency $\varphi \notin \Gamma$; and if $\varphi \notin \Gamma$ then $\varphi \in \Gamma$ since Γ is complete Σ-consistent.

6. Suppose $\varphi \land \psi \in \Gamma$. Since $(\varphi \land \psi) \to \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure, O.s., case (1). Similarly for $\psi \in \Gamma$. On the other hand, suppose both $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then deductive closure implies $(\varphi \land \psi) \in \Gamma$, since $\varphi \to (\psi \to (\varphi \land \psi))$ is a tautological instance.

7. Suppose $\varphi \lor \psi \in \Gamma$, and $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is complete Σ-consistent, $\lnot\varphi \in \Gamma$ and $\lnot\psi \in \Gamma$. Then $\lnot(\varphi \lor \psi) \in \Gamma$ since $\lnot\varphi \to (\lnot\psi \to \lnot(\varphi \lor \psi))$ is a tautological instance. This would mean that Γ is Σ-inconsistent, a contradiction.

8. Suppose $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$; then $\Gamma \vdash_\Sigma \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \to \psi \notin \Gamma$ then since Γ is complete Σ-consistent, $\lnot(\varphi \to \psi) \in \Gamma$. Since $\lnot(\varphi \to \psi) \to \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure. Since $\lnot(\varphi \to \psi) \to \lnot\psi$ is a tautological instance, $\lnot\psi \in \Gamma$. Then $\psi \notin \Gamma$ since Γ is Σ-consistent.

9. Suppose $\varphi \leftrightarrow \psi \in \Gamma$. If $\varphi \in \Gamma$, then $\psi \in \Gamma$, since $(\varphi \leftrightarrow \psi) \to (\varphi \to \psi)$ is a tautological instance. Similarly, if $\psi \in \Gamma$, then $\varphi \in \Gamma$. So either both $\varphi \in \Gamma$ and $\psi \in \Gamma$, or neither $\varphi \in \Gamma$ nor $\psi \in \Gamma$.

   Conversely, suppose $\varphi \to \psi \notin \Gamma$. Since Γ is complete Σ-consistent, $\lnot(\varphi \leftrightarrow \psi) \in \Gamma$. Since $\lnot(\varphi \leftrightarrow \psi) \to (\varphi \to \lnot\psi)$ is a tautological instance, if $\varphi \in \Gamma$ then $\lnot\psi \in \Gamma$, and since Γ is Σ-consistent, $\psi \notin \Gamma$. Similarly, if $\psi \in \Gamma$ then $\varphi \notin \Gamma$. So neither $\varphi \in \Gamma$ and $\psi \in \Gamma$, nor $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. □

Problema 4.1. Complete the proof of Proposição 4.2.

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4.3 Lindenbaum's Lemma

nml:com:lin:sec

Lindenbaum's Lemma establishes that every Σ-consistent set of fórmulas is contained in at least one *complete* Σ-consistent set. Our construction of the canonical model will show that for each complete Σ-consistent set Δ, there is a world in the canonical model where all and only the fórmulas in Δ are true. So Lindenbaum's Lemma guarantees that every Σ-consistent set is true at some world in the canonical model.

nml:com:lin:thm:lindenbaum

Teorema 4.3 (Lindenbaum's Lemma). *If Γ is Σ-consistent then there is a complete Σ-consistent set Δ extending Γ.*

Prova. Let $\varphi_0, \varphi_1, \ldots$ be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing p_0, and at each stage $n \geq 1$ list the finitely many formulas of length n using only variables among $p_0, \ldots, p_n$. We define sets of fórmulas Δ_n by induction on n, and we then set $\Delta = \bigcup_n \Delta_n$. We first put $\Delta_0 = \Gamma$. Supposing that Δ_n has been defined, we define Δ_{n+1} by:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is } \Sigma\text{-consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

Now let $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$.

We have to show that this definition actually yields a set Δ with the required properties, O.s., $\Gamma \subseteq \Delta$ and Δ is complete Σ-consistent.

It's obvious that $\Gamma \subseteq \Delta$, since $\Delta_0 \subseteq \Delta$ by construction, and $\Delta_0 = \Gamma$. In fact, $\Delta_n \subseteq \Delta$ for all n, since Δ is the union of all Δ_n. (Since in each step of the construction, we add a fórmula to the set already constructed, $\Delta_n \subseteq \Delta_{n+1}$, so since $\subseteq$ is transitive, $\Delta_n \subseteq \Delta_m$ whenever $n \leq m$.) At each stage of the construction, we either add φ_n or $\neg\varphi_n$, and every fórmula appears (at least once) in the list of all φ_n. So, for every φ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$, so Δ is complete by definition.

Finally, we have to show, that Δ is Σ-consistent. To do this, we show that (a) if Δ were Σ-inconsistent, then some Δ_n would be Σ-inconsistent, and (b) all Δ_n are Σ-consistent.

So suppose Δ were Σ-inconsistent. Then $\Delta \vdash_\Sigma \bot$, O.s., there are $\varphi_1, \ldots, \varphi_k \in \Delta$ such that $\Sigma \vdash \varphi_1 \to (\varphi_2 \to \cdots (\varphi_k \to \bot) \ldots)$. Since $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$, each $\varphi_i \in \Delta_{n_i}$ for some n_i. Let n be the largest of these. Since $n_i \leq n$, $\Delta_{n_i} \subseteq \Delta_n$. So, all φ_i are in some Δ_n. This would mean $\Delta_n \vdash_\Sigma \bot$, O.s., Δ_n is Σ-inconsistent.

To show that each Δ_n is Σ-consistent, we use a simple induction on n. $\Delta_0 = \Gamma$, and we assumed Γ was Σ-consistent. So the claim holds for $n = 0$. Now suppose it holds for n, O.s., Δ_n is Σ-consistent. Δ_{n+1} is either $\Delta_n \cup \{\varphi_n\}$ if that is Σ-consistent, otherwise it is $\Delta_n \cup \{\neg\varphi_n\}$. In the first case, Δ_{n+1} is clearly Σ-consistent. However, by Proposição 3.40(3), either $\Delta_n \cup \{\varphi_n\}$ or $\Delta_n \cup \{\neg\varphi_n\}$ is consistent, so Δ_{n+1} is consistent in the other case as well. □

Corolário 4.4. *$\Gamma \vdash_\Sigma \varphi$ if and only if $\varphi \in \Delta$ for each complete Σ-consistent set Δ extending Γ (including when $\Gamma = \emptyset$, in which case we get another characterization of the modal system Σ.)* nml:com:lin:cor:provability-characterization

Prova. Suppose $\Gamma \vdash_\Sigma \varphi$, and let Δ be any complete Σ-consistent set extending Γ. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so $\Delta \vdash_\Sigma \varphi$ (by monotonicity) and $\Delta \vdash_\Sigma \neg\varphi$ (by reflexivity), and so Δ is inconsistent. Conversely if $\Gamma \nvdash_\Sigma \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent, and by Lindenbaum's Lemma there is a complete consistent set Δ extending $\Gamma \cup \{\neg\varphi\}$. By consistency, $\varphi \notin \Delta$. □

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4.4 Modalities and Complete Consistent Sets

explanation When we construct a model $\mathfrak{M}^\Sigma$ whose set of worlds is given by the complete Σ-consistent sets Δ in some normal modal logic Σ, we will also need to define an accessibility relation R^Σ between such "worlds." We want it to be the case that the accessibility relation (and the assignment V^Σ) are defined in such a way that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. How should we do this? nml:com:mod:sec

Once the accessibility relation is defined, the definition of truth at a world ensures that $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\varphi$ iff $\mathfrak{M}^\Sigma, \Delta' \Vdash \varphi$ for all Δ' such that $R^\Sigma \Delta\Delta'$. The proof that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$ requires that this is true in particular for fórmulas starting with a modal operator, O.s., $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\varphi$ iff $\Box\varphi \in \Delta$. Combining this requirement with the definition of truth at a world for $\Box\varphi$ yields:

$$\Box\varphi \in \Delta \text{ iff } \varphi \in \Delta' \text{ for all } \Delta' \text{ with } R^\Sigma \Delta\Delta'$$

Consider the left-to-right direction: it says that if $\Box\varphi \in \Delta$, then $\varphi \in \Delta'$ for any φ and any Δ' with $R^\Sigma \Delta\Delta'$. If we stipulate that $R^\Sigma \Delta\Delta'$ iff $\varphi \in \Delta'$ for all $\Box\varphi \in \Delta$, then this holds. We can write the condition on the right of the "iff" more compactly as: $\{\varphi : \Box\varphi \in \Delta\} \subseteq \Delta'$.

So the question is: does this definition of R^Σ in fact guarantee that $\Box\varphi \in \Delta$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\varphi$? Does it also guarantee that $\Diamond\varphi \in \Delta$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\varphi$? The next few results will establish this.

Definição 4.5. If Γ is a set of fórmulas, let

$$\Box\Gamma = \{\Box\psi : \psi \in \Gamma\}$$
$$\Diamond\Gamma = \{\Diamond\psi : \psi \in \Gamma\}$$

and

$$\Box^{-1}\Gamma = \{\psi : \Box\psi \in \Gamma\}$$
$$\Diamond^{-1}\Gamma = \{\psi : \Diamond\psi \in \Gamma\}$$

In other words, $\Box\Gamma$ is Γ with $\Box$ in front of every fórmula in Γ; $\Box^{-1}\Gamma$ is all the $\Box$'ed fórmulas of Γ with the initial $\Box$'s removed. This definition is not terribly important on its own, but will simplify the notation considerably.

Note that $\Box\Box^{-1}\Gamma \subseteq \Gamma$:

$$\Box\Box^{-1}\Gamma = \{\Box\psi : \Box\psi \in \Gamma\}$$

O.s., it's just the set of all those fórmulas of Γ that start with $\Box$.

nml:com:mod: lem:box1

Lema 4.6. *If $\Gamma \vdash_\Sigma \varphi$ then $\Box\Gamma \vdash_\Sigma \Box\varphi$.*

Prova. If $\Gamma \vdash_\Sigma \varphi$ then there are $\psi_1, \ldots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots(\psi_n \to \varphi)\cdots)$. Since Σ is normal, by rule RK, $\Sigma \vdash \Box\psi_1 \to (\Box\psi_2 \to \cdots(\Box\psi_n \to \Box\varphi)\cdots)$, where obviously $\Box\psi_1, \ldots, \Box\psi_k \in \Box\Gamma$. Hence, by definition, $\Box\Gamma \vdash_\Sigma \Box\varphi$. $\Box$

nml:com:mod: lem:box2

Lema 4.7. *If $\Box^{-1}\Gamma \vdash_\Sigma \varphi$ then $\Gamma \vdash_\Sigma \Box\varphi$.*

Prova. Suppose $\Box^{-1}\Gamma \vdash_\Sigma \varphi$; then by Lema 4.6, $\Box\Box^{-1}\Gamma \vdash \Box\varphi$. But since $\Box\Box^{-1}\Gamma \subseteq \Gamma$, also $\Gamma \vdash_\Sigma \Box\varphi$ by monotonicity. $\Box$

nml:com:mod: prop:box

Proposição 4.8. *If Γ is complete Σ-consistent, then $\Box\varphi \in \Gamma$ if and only if for every complete Σ-consistent Δ such that $\Box^{-1}\Gamma \subseteq \Delta$, it holds that $\varphi \in \Delta$.*

Prova. Suppose Γ is complete Σ-consistent. The "only if" direction is easy: Suppose $\Box\varphi \in \Gamma$ and that $\Box^{-1}\Gamma \subseteq \Delta$. Since $\Box\varphi \in \Gamma$, $\varphi \in \Box^{-1}\Gamma \subseteq \Delta$, so $\varphi \in \Delta$.

For the "if" direction, we prove the contrapositive: Suppose $\Box\varphi \notin \Gamma$. Since Γ is complete Σ-consistent, it is deductively closed, and hence $\Gamma \nvdash_\Sigma \Box\varphi$. By Lema 4.7, $\Box^{-1}\Gamma \nvdash_\Sigma \varphi$. By Proposição 3.40(2), $\Box^{-1}\Gamma \cup \{\neg\varphi\}$ is Σ-consistent. By Lindenbaum's Lemma, there is a complete Σ-consistent set Δ such that $\Box^{-1}\Gamma \cup \{\neg\varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$. $\Box$

nml:com:mod: lem:box-iff-diamond

Lema 4.9. *Suppose Γ and Δ are complete Σ-consistent. Then $\Box^{-1}\Gamma \subseteq \Delta$ if and only if $\Diamond\Delta \subseteq \Gamma$.*

Prova. "Only if" direction: Assume $\Box^{-1}\Gamma \subseteq \Delta$ and suppose $\Diamond\varphi \in \Diamond\Delta$ (O.s., $\varphi \in \Delta$). In order to show $\Diamond\varphi \in \Gamma$, it suffices to show $\Box\neg\varphi \notin \Gamma$, for then by maximality, $\neg\Box\neg\varphi \in \Gamma$. Now, if $\Box\neg\varphi \in \Gamma$ then by hypothesis $\neg\varphi \in \Delta$, against the consistency of Δ (since $\varphi \in \Delta$). Hence $\Box\neg\varphi \notin \Gamma$, as required.

"If" direction: Assume $\Diamond\Delta \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box\varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg\varphi \in \Delta$ and so by hypothesis $\Diamond\neg\varphi \in \Gamma$. But in a normal modal logic $\Diamond\neg\varphi$ is equivalent to $\neg\Box\varphi$, and if the latter is in Γ, by consistency $\Box\varphi \notin \Gamma$, as required. $\Box$

nml:com:mod: prop:diamond

Proposição 4.10. *If Γ is complete Σ-consistent, then $\Diamond\varphi \in \Gamma$ if and only if for some complete Σ-consistent Δ such that $\Diamond\Delta \subseteq \Gamma$, it holds that $\varphi \in \Delta$.*

Prova. Suppose Γ is complete Σ-consistent. $\Diamond\varphi \in \Gamma$ iff $\neg\Box\neg\varphi \in \Gamma$ by DUAL and closure. $\neg\Box\neg\varphi \in \Gamma$ iff $\Box\neg\varphi \notin \Gamma$ by Proposição 4.2(5) since Γ is complete Σ-consistent. By Proposição 4.8, $\Box\neg\varphi \notin \Gamma$ iff, for some complete Σ-consistent Δ with $\Box^{-1}\Gamma \subseteq \Delta$, $\neg\varphi \notin \Delta$. Now consider any such Δ. By Lema 4.9, $\Box^{-1}\Gamma \subseteq \Delta$ iff $\Diamond\Delta \subseteq \Gamma$. Also, $\neg\varphi \notin \Delta$ iff $\varphi \in \Delta$ by Proposição 4.2(5). So $\Diamond\varphi \in \Gamma$ iff, for some complete Σ-consistent Δ with $\Diamond\Delta \subseteq \Gamma$, $\varphi \in \Delta$. □

Problema 4.2. Show that if Γ is complete Σ-consistent, then $\Diamond\varphi \in \Gamma$ if and only if there is a complete Σ-consistent Δ such that $\Box^{-1}\Gamma \subseteq \Delta$ and $\varphi \in \Delta$. *Do this without using Lema 4.9.*

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4.5 Canonical Models

The *canonical model* for a modal system Σ is a specific model $\mathfrak{M}^\Sigma$ in which the worlds are all complete Σ-consistent sets. Its accessibility relation R^Σ and valuation V^Σ are defined so as to guarantee that the fórmulas true at a world Δ are exactly the fórmulas making up Δ. nml:com:cmd:sec

Definição 4.11. Let Σ be a normal modal logic. The *canonical model* for Σ is $\mathfrak{M}^\Sigma = \langle W^\Sigma, R^\Sigma, V^\Sigma\rangle$, where:

1. $W^\Sigma = \{\Delta : \Delta \text{ is complete } \Sigma\text{-consistent}\}$.
2. $R^\Sigma\Delta\Delta'$ holds if and only if $\Box^{-1}\Delta \subseteq \Delta'$.
3. $V^\Sigma(p) = \{\Delta : p \in \Delta\}$.

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4.6 The Truth Lemma

The canonical model $\mathfrak{M}^\Sigma$ is defined in such a way that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. For variáveis proposicionais, the definition of V^Σ yields this directly. We have to verify that the equivalence holds for all fórmulas, however. We do this by induction. The inductive step involves proving the equivalence for fórmulas involving propositional operators (where we have to use Proposição 4.2) and the modal operators (where we invoke the results of Seção 4.4). nml:com:tru:sec

Proposição 4.12 (Truth Lemma). *For every fórmula φ, $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ if and only if $\varphi \in \Delta$.* *nml:com:tru:prop:truthlemma*

Prova. By induction on φ.

1. $\varphi \equiv \bot$: $\mathfrak{M}^\Sigma, \Delta \nVdash \bot$ by Definição 1.6, and $\bot \notin \Delta$ by Proposição 4.2(3).
2. $\varphi \equiv \top$: $\mathfrak{M}^\Sigma, \Delta \Vdash \top$ by Definição 1.6, and $\top \in \Delta$ by Proposição 4.2(4).

3. $\varphi \equiv p$: $\mathfrak{M}^{\Sigma}, \Delta \Vdash p$ iff $\Delta \in V^{\Sigma}(p)$ by Definição 1.6. Also, $\Delta \in V^{\Sigma}(p)$ iff $p \in \Delta$ by definition of V^{Σ}.

4. $\varphi \equiv \neg\psi$: $\mathfrak{M}^{\Sigma}, \Delta \Vdash \neg\psi$ iff $\mathfrak{M}^{\Sigma}, \Delta \nVdash \psi$ (Definição 1.6) iff $\psi \notin \Delta$ (by inductive hypothesis) iff $\neg\psi \in \Delta$ (by Proposição 4.2(5)).

5. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{M}^{\Sigma}, \Delta \Vdash \psi \wedge \chi$ iff $\mathfrak{M}^{\Sigma}, \Delta \Vdash \psi$ and $\mathfrak{M}^{\Sigma}, \Delta \Vdash \chi$ (by Definição 1.6) iff $\psi \in \Delta$ and $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \wedge \chi \in \Delta$ (by Proposição 4.2(6)).

6. $\varphi \equiv \psi \vee \chi$: $\mathfrak{M}^{\Sigma}, \Delta \Vdash \psi \vee \chi$ iff $\mathfrak{M}^{\Sigma}, \Delta \Vdash \psi$ or $\mathfrak{M}^{\Sigma}, \Delta \Vdash \chi$ (by Definição 1.6) iff $\psi \in \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \vee \chi \in \Delta$ (by Proposição 4.2(7)).

7. $\varphi \equiv \psi \rightarrow \chi$: $\mathfrak{M}^{\Sigma}, \Delta \Vdash \psi \rightarrow \chi$ iff $\mathfrak{M}^{\Sigma}, \Delta \nVdash \psi$ or $\mathfrak{M}^{\Sigma}, \Delta \Vdash \chi$ (by Definição 1.6) iff $\psi \notin \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \rightarrow \chi \in \Delta$ (by Proposição 4.2(8)).

8. $\varphi \equiv \psi \leftrightarrow \chi$: $\mathfrak{M}^{\Sigma}, \Delta \Vdash \psi \leftrightarrow \chi$ iff either $\mathfrak{M}^{\Sigma}, \Delta \Vdash \psi$ and $\mathfrak{M}^{\Sigma}, \Delta \Vdash \chi$ or $\mathfrak{M}^{\Sigma}, \Delta \nVdash \psi$ and $\mathfrak{M}^{\Sigma}, \Delta \nVdash \chi$ (by Definição 1.6) iff either $\psi \in \Delta$ and $\chi \in \Delta$ or $\psi \notin \Delta$ and $\chi \notin \Delta$ (by inductive hypothesis) iff $\psi \leftrightarrow \chi \in \Delta$ (by Proposição 4.2(9)).

9. $\varphi \equiv \Box\psi$: First suppose that $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Box\psi$. By Definição 1.6, for every Δ' such that $R^{\Sigma}\Delta\Delta'$, $\mathfrak{M}^{\Sigma}, \Delta' \Vdash \psi$. By inductive hypothesis, for every Δ' such that $R^{\Sigma}\Delta\Delta'$, $\psi \in \Delta'$. By definition of R^{Σ}, for every Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$, $\psi \in \Delta'$. By Proposição 4.8, $\Box\psi \in \Delta$.

   Now assume $\Box\psi \in \Delta$. Let $\Delta' \in W^{\Sigma}$ be such that $R^{\Sigma}\Delta\Delta'$, O.s., $\Box^{-1}\Delta \subseteq \Delta'$. Since $\Box\psi \in \Delta$, $\psi \in \Box^{-1}\Delta$. Consequently, $\psi \in \Delta'$. By inductive hypothesis, $\mathfrak{M}^{\Sigma}, \Delta' \Vdash \psi$. Since Δ' is arbitrary with $R^{\Sigma}\Delta\Delta'$, for all $\Delta' \in W^{\Sigma}$ such that $R^{\Sigma}\Delta\Delta'$, $\mathfrak{M}^{\Sigma}, \Delta' \Vdash \psi$. By Definição 1.6, $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Box\psi$.

10. $\varphi \equiv \Diamond\psi$: First suppose that $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Diamond\psi$. By Definição 1.6, for some Δ' such that $R^{\Sigma}\Delta\Delta'$, $\mathfrak{M}^{\Sigma}, \Delta' \Vdash \psi$. By inductive hypothesis, for some Δ' such that $R^{\Sigma}\Delta\Delta'$, $\psi \in \Delta'$. By definition of R^{Σ}, for some Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$, $\psi \in \Delta'$. By Proposição 4.10, for some Δ' such that $\Diamond\Delta' \subseteq \Delta$, $\psi \in \Delta'$. Since $\psi \in \Delta'$, $\Diamond\psi \in \Diamond\Delta'$, so $\Diamond\psi \in \Delta$.

    Now assume $\Diamond\psi \in \Delta$. By Proposição 4.10, there is a complete Σ-consistent $\Delta' \in W^{\Sigma}$ such that $\Diamond\Delta' \subseteq \Delta$ and $\psi \in \Delta'$. By Lema 4.9, there is a $\Delta' \in W^{\Sigma}$ such that $\Box^{-1}\Delta \subseteq \Delta'$, and $\psi \in \Delta'$. By definition of R^{Σ}, $R^{\Sigma}\Delta\Delta'$, so there is a $\Delta' \in W^{\Sigma}$ such that $R^{\Sigma}\Delta\Delta'$ and $\psi \in \Delta'$. By Definição 1.6, $\mathfrak{M}^{\Sigma}, \Delta \Vdash \Diamond\psi$. □

Problema 4.3. Complete the proof of Proposição 4.12.

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4.7 Determination and Completeness for K

We are now prepared to use the canonical model to establish completeness. Completeness follows from the fact that the fórmulas true in the canonical model for Σ are exactly the Σ-derivável ones. Models with this property are said to *determine* Σ. nml:com:cmk:sec

Definição 4.13. A model $\mathfrak{M}$ *determines* a normal modal logic Σ precisely when $\mathfrak{M} \Vdash \varphi$ if and only if $\Sigma \vdash \varphi$, for all fórmulas φ.

Teorema 4.14 (Determination). *$\mathfrak{M}^\Sigma \Vdash \varphi$ if and only if $\Sigma \vdash \varphi$.* *nml:com:cmk:thm:determination*

Prova. If $\mathfrak{M}^\Sigma \Vdash \varphi$, then for every complete Σ-consistent Δ, we have $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$. Hence, by the Truth Lemma, $\varphi \in \Delta$ for every complete Σ-consistent Δ, whence by Corolário 4.4 (with $\Gamma = \emptyset$), $\Sigma \vdash \varphi$.

Conversely, if $\Sigma \vdash \varphi$ then by Proposição 4.2(1), every complete Σ-consistent Δ contains φ, and hence by the Truth Lemma, $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ for every $\Delta \in W^\Sigma$, O.s., $\mathfrak{M}^\Sigma \Vdash \varphi$. □

Since the canonical model for **K** determines **K**, we immediately have completeness of **K** as a corollary:

Corolário 4.15. *The basic modal logic* **K** *is complete with respect to the class of all models, O.s., if $\vDash \varphi$ then* $\mathbf{K} \vdash \varphi$. *nml:com:cmk:cor:Kcomplete*

Prova. Contrapositively, if $\mathbf{K} \nvdash \varphi$ then by Determination $\mathfrak{M}^{\mathbf{K}} \nVdash \varphi$ and hence φ is not valid. □

For the general case of completeness of a system Σ with respect to a class of models, p.ex., of **KTB4** with respect to the class of reflexive, symmetric, transitive models, determination alone is not enough. We must also show that the canonical model for the system Σ is a member of the class, which does not follow obviously from the canonical model construction—nor is it always true!

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4.8 Frame Completeness

The completeness theorem for **K** can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property. nml:com:fra:sec

Teorema 4.16. *If a normal modal logic Σ contains one of the fórmulas on the left-hand side of Tabela 4.1, then the canonical model for Σ has the corresponding property on the right-hand side.* *nml:com:fra:thm:completeframeprops*

If Σ contains ...	*... the canonical model for Σ is:*
D: $\Box\varphi \to \Diamond\varphi$	serial;
T: $\Box\varphi \to \varphi$	reflexive;
B: $\varphi \to \Box\Diamond\varphi$	symmetric;
4: $\Box\varphi \to \Box\Box\varphi$	transitive;
5: $\Diamond\varphi \to \Box\Diamond\varphi$	euclidean.

Tabela 4.1: Basic correspondence facts.

nml:com:fra:
tab:correspondencetable

Prova. We take each of these up in turn.

Suppose Σ contains D, and let $\Delta \in W^\Sigma$; we need to show that there is a Δ' such that $R^\Sigma\Delta\Delta'$. It suffices to show that $\Box^{-1}\Delta$ is Σ-consistent, for then by Lindenbaum's Lemma, there is a complete Σ-consistent set $\Delta' \supseteq \Box^{-1}\Delta$, and by definition of R^Σ we have $R^\Sigma\Delta\Delta'$. So, suppose for contradiction that $\Box^{-1}\Delta$ is *not* Σ-consistent, O.s., $\Box^{-1}\Delta \vdash_\Sigma \bot$. By Lema 4.7, $\Delta \vdash_\Sigma \Box\bot$, and since Σ contains D, also $\Delta \vdash_\Sigma \Diamond\bot$. But Σ is normal, so $\Sigma \vdash \neg\Diamond\bot$ (Proposição 3.7), whence also $\Delta \vdash_\Sigma \neg\Diamond\bot$, against the consistency of Δ.

Now suppose Σ contains T, and let $\Delta \in W^\Sigma$. We want to show $R^\Sigma\Delta\Delta$, O.s., $\Box^{-1}\Delta \subseteq \Delta$. But if $\Box\varphi \in \Delta$ then by T also $\varphi \in \Delta$, as desired.

Now suppose Σ contains B, and suppose $R^\Sigma\Delta\Delta'$ for Δ, $\Delta' \in W^\Sigma$. We need to show that $R^\Sigma\Delta'\Delta$, O.s., $\Box^{-1}\Delta' \subseteq \Delta$. By Lema 4.9, this is equivalent to $\Diamond\Delta \subseteq \Delta'$. So suppose $\varphi \in \Delta$. By B, also $\Box\Diamond\varphi \in \Delta$. By the hypothesis that $R^\Sigma\Delta\Delta'$, we have that $\Box^{-1}\Delta \subseteq \Delta'$, and hence $\Diamond\varphi \in \Delta'$, as required.

Now suppose Σ contains 4, and suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_2\Delta_3$. We need to show $R^\Sigma\Delta_1\Delta_3$. From the hypothesis we have both $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and $\Box^{-1}\Delta_2 \subseteq \Delta_3$. In order to show $R^\Sigma\Delta_1\Delta_3$ it suffices to show $\Box^{-1}\Delta_1 \subseteq \Delta_3$. So let $\psi \in \Box^{-1}\Delta_1$, O.s., $\Box\psi \in \Delta_1$. By 4, also $\Box\Box\psi \in \Delta_1$ and by hypothesis we get, first, that $\Box\psi \in \Delta_2$ and, second, that $\psi \in \Delta_3$, as desired.

Now suppose Σ contains 5, suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$. We need to show $R^\Sigma\Delta_2\Delta_3$. The first hypothesis gives $\Box^{-1}\Delta_1 \subseteq \Delta_2$, and the second hypothesis is equivalent to $\Diamond\Delta_3 \subseteq \Delta_1$, by Lema 4.9. To show $R^\Sigma\Delta_2\Delta_3$, by Lema 4.9, it suffices to show $\Diamond\Delta_3 \subseteq \Delta_2$. So let $\Diamond\varphi \in \Diamond\Delta_3$, O.s., $\varphi \in \Delta_3$. By the second hypothesis $\Diamond\varphi \in \Delta_1$ and by 5, $\Box\Diamond\varphi \in \Delta_1$ as well. But now the first hypothesis gives $\Diamond\varphi \in \Delta_2$, as desired. □

As a corollary we obtain completeness results for a number of systems. For instance, we know that **S5** = **KT5** = **KTB4** is complete with respect to the class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

nml:com:fra:
thm:generaldet

Teorema 4.17. *Let $\mathcal{C}_\mathrm{D}$, $\mathcal{C}_\mathrm{T}$, $\mathcal{C}_\mathrm{B}$, $\mathcal{C}_4$, and $\mathcal{C}_5$ be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\varphi_1, \dots, \varphi_n$ among* D, T, B, 4, *and* 5, *the system $\mathbf{K}\varphi_1 \dots \varphi_n$ is determined by the class of models $\mathcal{C} = \mathcal{C}_{\varphi_1} \cap \dots \cap \mathcal{C}_{\varphi_n}$.*

Proposição 4.18. *Let Σ be a normal modal logic; then:*

1. *If Σ contains the schema $\Diamond\varphi \to \Box\varphi$ then the canonical model for Σ is partially functional.* nml:com:fra:prop:anotherfive-a

2. *If Σ contains the schema $\Diamond\varphi \leftrightarrow \Box\varphi$ then the canonical model for Σ is functional.*

3. *If Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ then the canonical model for Σ is weakly dense.*

(see Tabela 2.2 for definitions of these frame properties).

Prova. 1. Suppose that Σ contains the schema $\Diamond\varphi \to \Box\varphi$, to show that R^Σ is partially functional we need to prove that for any Δ_1, Δ_2, $\Delta_3 \in W^\Sigma$, if $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$ then $\Delta_2 = \Delta_3$. Since $R^\Sigma\Delta_1\Delta_2$ we have $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and since $R^\Sigma\Delta_1\Delta_3$ also $\Box^{-1}\Delta_1 \subseteq \Delta_3$. The identity $\Delta_2 = \Delta_3$ will follow if we can establish the two inclusions $\Delta_2 \subseteq \Delta_3$ and $\Delta_3 \subseteq \Delta_2$. For the first inclusion, let $\varphi \in \Delta_2$; then $\Diamond\varphi \in \Delta_1$, and by the schema and deductive closure of Δ_1 also $\Box\varphi \in \Delta_1$, whence by the hypothesis that $R^\Sigma\Delta_1\Delta_3$, $\varphi \in \Delta_3$. The second inclusion is similar.

2. This follows immediately from part (1) and the seriality proof in Teorema 4.16.

3. Suppose Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ and to show that R^Σ is weakly dense, let $R^\Sigma\Delta_1\Delta_2$. We need to show that there is a complete Σ-consistent set Δ_3 such that $R^\Sigma\Delta_1\Delta_3$ and $R^\Sigma\Delta_3\Delta_2$. Let:

$$\Gamma = \Box^{-1}\Delta_1 \cup \Diamond\Delta_2.$$

It suffices to show that Γ is Σ-consistent, for then by Lindenbaum's Lemma it can be extended to a complete Σ-consistent set Δ_3 such that $\Box^{-1}\Delta_1 \subseteq \Delta_3$ and $\Diamond\Delta_2 \subseteq \Delta_3$, O.s., $R^\Sigma\Delta_1\Delta_3$ and $R^\Sigma\Delta_3\Delta_2$ (by Lema 4.9).

Suppose for contradiction that Γ is not consistent. Then there are fórmulas $\Box\varphi_1, \ldots, \Box\varphi_n \in \Delta_1$ and $\psi_1, \ldots, \psi_m \in \Delta_2$ such that

$$\varphi_1, \ldots, \varphi_n, \Diamond\psi_1, \ldots, \Diamond\psi_m \vdash_\Sigma \bot.$$

Since $\Diamond(\psi_1 \land \cdots \land \psi_m) \to (\Diamond\psi_1 \land \cdots \land \Diamond\psi_m)$ is derivável in every normal modal logic, we argue as follows, contradicting the consistency of Δ_2:

$$
\begin{aligned}
\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m &\vdash_\Sigma \bot \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma (\Diamond\psi_1 \land \dots \land \Diamond\psi_m) \to \bot \\
&\quad \text{by the deduction theorem} \\
&\quad \text{Proposição 3.37(4), and TAUT} \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma \Diamond(\psi_1 \land \dots \land \psi_m) \to \bot \\
&\quad \text{since } \Sigma \text{ is normal} \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma \lnot\Diamond(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by PL} \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma \Box\lnot(\psi_1 \land \dots \land \psi_m) \\
&\quad \Box\lnot \text{ for } \lnot\Diamond \\
\Box\varphi_1, \dots, \Box\varphi_n &\vdash_\Sigma \Box\Box\lnot(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by Lema 4.6} \\
\Box\varphi_1, \dots, \Box\varphi_n &\vdash_\Sigma \Box\lnot(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by schema } \Box\Box\varphi \to \Box\varphi \\
\Delta_1 &\vdash_\Sigma \Box\lnot(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by monotonicity, Proposição 3.37(1)} \\
\Box\lnot(\psi_1 \land \dots \land \psi_m) &\in \Delta_1 \\
&\quad \text{by deductive closure;} \\
\lnot(\psi_1 \land \dots \land \psi_m) &\in \Delta_2 \\
&\quad \text{since } R^\Sigma\Delta_1\Delta_2.
\end{aligned}
$$

□

On the strength of these examples, one might think that every system Σ of modal logic is *complete*, in the sense that it proves every formula which is valid in every frame in which every theorem of Σ is valid. Unfortunately, there are many systems that are not complete in this sense.

Capítulo 5

Filtrations and Decidability

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5.1 Introduction

One important question about a logic is always whether it is decidable, O.s., if there is an effective procedure which will answer the question "is this fórmula valid." Propositional logic is decidable: we can effectively test if a fórmula is a tautology by constructing a truth table, and for a given fórmula, the truth table is finite. But we can't obviously test if a modal formula is true in all models, for there are infinitely many of them. We can list all the finite models relevant to a given formula, since only the assignment of subsets of worlds to variáveis proposicionais which actually occur in the fórmula are relevant. If the accessibility relation is fixed, the possible different assignments $V(p)$ are just all the subsets of W, and if $|W| = n$ there are 2^n of those. If our fórmula φ contains m variáveis proposicionais there are then 2^{nm} different models with n worlds. For each one, we can test if φ is true at all worlds, simply by computing the truth value of φ in each. Of course, we also have to check all possible accessibility relations, but there are only finitely many relations on n worlds as well (specifically, the number of subsets of $W \times W$, O.s., 2^{n^2}. nml:fil:int:sec

If we are not interested in the logic **K**, but a logic defined by some class of models (p.ex., the reflexive transitive models), we also have to be able to test if the accessibility relation is of the right kind. We can do that whenever the frames we are interested in are definable by modal fórmulas (p.ex., by testing if T and 4 valid in the frame). So, the idea would be to run through all the finite frames, test each one if it is a frame in the class we're interested in, then list all the possible models on that frame and test if φ is true in each. If not, stop: φ is not valid in the class of models of interest.

There is a problem with this idea: we don't know when, if ever, we can stop looking. If the fórmula has a finite countermodel, our procedure will find it. But if it has no finite countermodel, we won't get an answer. The fórmula may be valid (no countermodels at all), or it may have only an infinite countermodel, which we'll never look at. This problem can be overcome if we can show that every fórmula that has a countermodel has a finite countermodel. If this is the case we say the logic has the *finite model property.*

But how would we show that a logic has the finite model property? One way of doing this would be to find a way to turn an infinite (counter)model of φ into a finite one. If that can be done, then whenever there is a model in which φ is not true, then the resulting finite model also makes φ not true. That finite model will show up on our list of all finite models, and we will eventually determine, for every formula that is not valid, that it isn't. Our procedure won't terminate if the formula is valid. If we can show in addition that there is some maximum size that the finite model our procedure provides can have, and that this maximum size depends only on the fórmula φ, we will have a size up to which we have to test finite models in our search for countermodels. If we haven't found a countermodel by then, there are none. Then our procedure will, in fact, decide the question "is φ valid?" for any fórmula φ.

A strategy that often works for turning infinite structures into finite structures is that of "identifying" membros of the structure which behave the same way in relevant respects. If there are infinitely many worlds in $\mathfrak{M}$ that behave the same in relevant respects, then we may be able to collect *all* worlds in finitely many (possibly infinite) "classes" of such worlds. In other words, we should partition the set of worlds in the right way, O.s., in such a way that each partition contains infinitely many worlds, but there are only finitely many partitions. Then we define a new model $\mathfrak{M}^*$ where the worlds are the partitions. Finitely many partitions in the old model give us finitely many worlds in the new model, O.s., a finite model. Let's call the partition a world w is in $[w]$. We'll want it to be the case that $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}^*, [w] \Vdash \varphi$, since we want the new model to be a countermodel to φ if the old one was. This requires that we define the partition, as well as the accessibility relation of $\mathfrak{M}^*$ in the right way.

To see how this would go, first imagine we have no accessibility relation. $\mathfrak{M}, w \Vdash \Box\psi$ iff for some $v \in W$, $\mathfrak{M}, v \Vdash \Box\psi$, and the same for $\mathfrak{M}^*$, except with $[w]$ and $[v]$. As a first idea, let's say that two worlds u and v are equivalent (belong to the same partition) if they agree on all variáveis proposicionais in $\mathfrak{M}$, O.s., $\mathfrak{M}, u \Vdash p$ iff $\mathfrak{M}, v \Vdash p$. Let $V^*(p) = \{[w] : \mathfrak{M}, w \Vdash p\}$. Our aim is to show that $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}^*, [w] \Vdash \varphi$. Obviously, we'd prove this by induction: The base case would be $\varphi \equiv p$. First suppose $\mathfrak{M}, w \Vdash p$. Then $[w] \in V^*$ by definition, so $\mathfrak{M}^*, [w] \Vdash p$. Now suppose that $\mathfrak{M}^*, [w] \Vdash p$. That means that $[w] \in V^*(p)$, O.s., for some v equivalent to w, $\mathfrak{M}, v \Vdash p$. But "w equivalent to v" means "w and v make all the same variáveis proposicionais true," so $\mathfrak{M}, w \Vdash p$. Now for the inductive step, p.ex., $\varphi \equiv \neg\psi$. Then $\mathfrak{M}, w \Vdash \neg\psi$ iff $\mathfrak{M}, w \nVdash \psi$ iff $\mathfrak{M}^*, [w] \nVdash \psi$ (by inductive hypothesis) iff $\mathfrak{M}^*, [w] \Vdash \neg\psi$. Similarly for the other non-modal operators. It also works for $\Box$: suppose $\mathfrak{M}^*, [w] \Vdash \Box\psi$. That means that for every $[u]$, $\mathfrak{M}^*, [u] \Vdash \psi$. By inductive hypothesis, for every u, $\mathfrak{M}, u \Vdash \psi$. Consequently, $\mathfrak{M}, w \Vdash \Box\psi$.

In the general case, where we have to also define the accessibility relation for $\mathfrak{M}^*$, things are more complicated. We'll call a model $\mathfrak{M}^*$ a *filtration* if its accessibility relation R^* satisfies the conditions required to make the inductive proof above go through. Then any filtration $\mathfrak{M}^*$ will make φ true at $[w]$ iff $\mathfrak{M}$ makes φ true at w. However, now we also have to show that there *are* filtrations, O.s., we can define R^* so that it satisfies the required conditions. In order for this to work, however, we have to require that worlds u, v count as equivalent not just when they agree on all variáveis proposicionais, but on all sub-fórmulas of φ. Since φ has only finitely many sub-fórmulas, this will still guarantee that the filtration is finite. There is not just one way to define a filtration, and in order to make sure that the accessibility relation of the filtration satisfies the required properties (p.ex., reflexive, transitive, etc.) we have to be inventive with the definition of R^*.

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5.2 Preliminaries

Filtrations allow us to establish the decidability of our systems of modal logic by showing that they have the *finite model property*, O.s., that any fórmula that is true (false) in a model is also true (false) in a *finite* model. Filtrations are defined relative to sets of fórmulas which are closed under subformulas. nml:fil:pre:sec

Definição 5.1. A set Γ of fórmulas is *closed under subformulas* if it contains every subformula of a fórmula in Γ. Further, Γ is *modally closed* if it is closed under subformulas and moreover $\varphi \in \Gamma$ implies $\Box\varphi, \Diamond\varphi \in \Gamma$. nml:fil:pre:defn:modallyclosed

For instance, given a fórmula φ, the set of all its sub-fórmulas is closed under sub-fórmulas. When we're defining a filtration of a model through the set of sub-fórmulas of φ, it will have the property we're after: it makes φ true (false) iff the original model does.

The set of worlds of a filtration of $\mathfrak{M}$ through Γ is defined as the set of all equivalence classes of the following equivalence relation.

Definição 5.2. Let $\mathfrak{M} = \langle W, R, V\rangle$ and suppose Γ is closed under sub-fórmulas. Define a relation $\equiv$ on W to hold of any two worlds that make the same fórmulas from Γ true, O.s.:

$$u \equiv v \quad \text{if and only if} \quad \forall\varphi \in \Gamma : \mathfrak{M}, u \Vdash \varphi \Leftrightarrow \mathfrak{M}, v \Vdash \varphi.$$

The equivalence class $[w]_\equiv$ of a world w, or $[w]$ for short, is the set of all worlds $\equiv$-equivalent to w:

$$[w] = \{v : v \equiv w\}.$$

Proposição 5.3. *Given $\mathfrak{M}$ and Γ, $\equiv$ as defined above is an equivalence relation, O.s., it is reflexive, symmetric, and transitive.*

Prova. The relation $\equiv$ is reflexive, since w makes exactly the same fórmulas from Γ true as itself. It is symmetric since if u makes the same fórmulas from Γ true as v, the same holds for v and u. It is also transitive, since if u makes the same fórmulas from Γ true as v, and v as w, then u makes the same fórmulas from Γ true as w. □

The relation $\equiv$, like any equivalence relation, divides W into *partitions*, O.s., subsets of W which are pairwise disjoint, and together cover all of W. Every $w \in W$ is um membro of one of the partitions, namely of $[w]$, since $w \equiv w$. So the partitions $[w]$ cover all of W. They are pairwise disjoint, for if $u \in [w]$ and $u \in [v]$, then $u \equiv w$ and $u \equiv v$, and by symmetry and transitivity, $w \equiv v$, and so $[w] = [v]$.

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5.3 Filtrations

nml:fil:fil:sec Rather than define "the" filtration of $\mathfrak{M}$ through Γ, we define when a model $\mathfrak{M}^*$ counts as a filtration of $\mathfrak{M}$. All filtrations have the same set of worlds W^* and the same valuation V^*. But different filtrations may have different accessibility relations R^*. To count as a filtration, R^* has to satisfy a number of conditions, however. These conditions are exactly what we'll require to prove the main result, namely that $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}^*, [w] \Vdash \varphi$, provided $\varphi \in \Gamma$.

nml:fil:fil:defn:filtration **Definição 5.4.** Let Γ be closed under subformulas and $\mathfrak{M} = \langle W, R, V\rangle$. A *filtration of* $\mathfrak{M}$ *through* Γ is any model $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$, where:

1. $W^* = \{[w] : w \in W\}$;
2. nml:fil:fil:defn:filtration-R For any $u, v \in W$:
   a) nml:fil:fil:defn:filtration-R1 If Ruv then $R^*[u][v]$;
   b) nml:fil:fil:defn:filtration-R2 If $R^*[u][v]$ then for any $\Box\varphi \in \Gamma$, if $\mathfrak{M}, u \Vdash \Box\varphi$ then $\mathfrak{M}, v \Vdash \varphi$;
   c) nml:fil:fil:defn:filtration-R3 If $R^*[u][v]$ then for any $\Diamond\varphi \in \Gamma$, if $\mathfrak{M}, v \Vdash \varphi$ then $\mathfrak{M}, u \Vdash \Diamond\varphi$.
3. $V^*(p) = \{[u] : u \in V(p)\}$.

It's worthwhile thinking about what $V^*(p)$ is: the set consisting of the equivalence classes $[w]$ of all worlds w where p is true in $\mathfrak{M}$. On the one hand, if $w \in V(p)$, then $[w] \in V^*(p)$ by that definition. However, it is not necessarily the case that if $[w] \in V^*(p)$, then $w \in V(p)$. If $[w] \in V^*(p)$ we are only guaranteed that $[w] = [u]$ for *some* $u \in V(p)$. Of course, $[w] = [u]$ means that $w \equiv u$. So, when $[w] \in V^*(p)$ we can (only) conclude that $w \equiv u$ for some $u \in V(p)$.

nml:fil:fil:thm:filtrations **Teorema 5.5.** *If $\mathfrak{M}^*$ is a filtration of $\mathfrak{M}$ through Γ, then for every $\varphi \in \Gamma$ and $w \in W$, we have $\mathfrak{M}, w \Vdash \varphi$ if and only if $\mathfrak{M}^*, [w] \Vdash \varphi$.*

Prova. By induction on φ, using the fact that Γ is closed under subformulas. Since $\varphi \in \Gamma$ and Γ is closed under sub-fórmulas, all sub-fórmulas of φ are also $\in \Gamma$. Hence in each inductive step, the induction hypothesis applies to the sub-fórmulas of φ.

1. $\varphi \equiv \bot$: Neither $\mathfrak{M}, w \Vdash \varphi$ nor $\mathfrak{M}^*, [w] \Vdash \varphi$.

2. $\varphi \equiv \top$: Both $\mathfrak{M}, w \Vdash \varphi$ and $\mathfrak{M}^*, [w] \Vdash \varphi$.

3. $\varphi \equiv p$: The left-to-right direction is immediate, as $\mathfrak{M}, w \Vdash \varphi$ only if $w \in V(p)$, which implies $[w] \in V^*(p)$, O.s., $\mathfrak{M}^*, [w] \Vdash \varphi$. Conversely, suppose $\mathfrak{M}^*, [w] \Vdash \varphi$, O.s., $[w] \in V^*(p)$. Then for some $v \in V(p)$, $w \equiv v$. Of course then also $\mathfrak{M}, v \Vdash p$. Since $w \equiv v$, w and v make the same fórmulas from Γ true. Since by assumption $p \in \Gamma$ and $\mathfrak{M}, v \Vdash p$, $\mathfrak{M}, w \Vdash \varphi$.

4. $\varphi \equiv \neg\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \nVdash \psi$. By induction hypothesis, $\mathfrak{M}, w \nVdash \psi$ iff $\mathfrak{M}^*, [w] \nVdash \psi$. Finally, $\mathfrak{M}^*, [w] \nVdash \psi$ iff $\mathfrak{M}^*, [w] \Vdash \varphi$.

5. $\varphi \equiv (\psi \land \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$. By induction hypothesis, $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}^*, [w] \Vdash \psi$, and $\mathfrak{M}, w \Vdash \chi$ iff $\mathfrak{M}^*, [w] \Vdash \chi$. And $\mathfrak{M}^*, [w] \Vdash \varphi$ iff $\mathfrak{M}^*, [w] \Vdash \psi$ and $\mathfrak{M}^*, [w] \Vdash \chi$.

6. $\varphi \equiv (\psi \lor \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$. By induction hypothesis, $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}^*, [w] \Vdash \psi$, and $\mathfrak{M}, w \Vdash \chi$ iff $\mathfrak{M}^*, [w] \Vdash \chi$. And $\mathfrak{M}^*, [w] \Vdash \varphi$ iff $\mathfrak{M}^*, [w] \Vdash \psi$ or $\mathfrak{M}^*, [w] \Vdash \chi$.

7. $\varphi \equiv (\psi \to \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \nVdash \psi$ or $\mathfrak{M}, w \Vdash \chi$. By induction hypothesis, $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}^*, [w] \Vdash \psi$, and $\mathfrak{M}, w \Vdash \chi$ iff $\mathfrak{M}^*, [w] \Vdash \chi$. And $\mathfrak{M}^*, [w] \Vdash \varphi$ iff $\mathfrak{M}^*, [w] \nVdash \psi$ or $\mathfrak{M}^*, [w] \Vdash \chi$.

8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$, or $\mathfrak{M}, w \nVdash \psi$ and $\mathfrak{M}, w \nVdash \chi$. By induction hypothesis, $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}^*, [w] \Vdash \psi$, and $\mathfrak{M}, w \Vdash \chi$ iff $\mathfrak{M}^*, [w] \Vdash \chi$. And $\mathfrak{M}^*, [w] \Vdash \varphi$ iff $\mathfrak{M}^*, [w] \Vdash \psi$ and $\mathfrak{M}^*, [w] \Vdash \chi$, or $\mathfrak{M}^*, [w] \nVdash \psi$ and $\mathfrak{M}^*, [w] \nVdash \chi$.

9. $\varphi \equiv \Box\psi$: Suppose $\mathfrak{M}, w \Vdash \varphi$; to show that $\mathfrak{M}^*, [w] \Vdash \varphi$, let v be such that $R^*[w][v]$. From Definição 5.4(2b), we have that $\mathfrak{M}, v \Vdash \psi$, and by inductive hypothesis $\mathfrak{M}^*, [v] \Vdash \psi$. Since v was arbitrary, $\mathfrak{M}^*, [w] \Vdash \varphi$ follows.

   Conversely, suppose $\mathfrak{M}^*, [w] \Vdash \varphi$ and let v be arbitrary such that Rwv. From Definição 5.4(2a), we have $R^*[w][v]$, so that $\mathfrak{M}^*, [v] \Vdash \psi$; by inductive hypothesis $\mathfrak{M}, v \Vdash \psi$, and since v was arbitrary, $\mathfrak{M}, w \Vdash \varphi$.

10. $\varphi \equiv \Diamond\psi$: Suppose $\mathfrak{M}, w \Vdash \varphi$. Then for some $v \in W$, Rwv and $\mathfrak{M}, v \Vdash \psi$. By inductive hypothesis $\mathfrak{M}^*, [v] \Vdash \psi$, and by Definição 5.4(2a), we have $R^*[w][v]$. Thus, $\mathfrak{M}^*, [w] \Vdash \varphi$.

    Now suppose $\mathfrak{M}^*, [w] \Vdash \varphi$. Then for some $[v] \in W^*$ with $R^*[w][v]$, $\mathfrak{M}^*, [v] \Vdash \psi$. By inductive hypothesis $\mathfrak{M}, v \Vdash \psi$. By Definição 5.4(2c), we have that $\mathfrak{M}, w \Vdash \varphi$. □

Problema 5.1. Complete the proof of Teorema 5.5

What holds for truth at worlds in a model also holds for truth in a model and validity in a class of models.

Corolário 5.6. *Let Γ be closed under subformulas. Then:*

1. *If $\mathfrak{M}^*$ is a filtration of $\mathfrak{M}$ through Γ then for any $\varphi \in \Gamma$: $\mathfrak{M} \Vdash \varphi$ if and only if $\mathfrak{M}^* \Vdash \varphi$.*

2. *If $\mathcal{C}$ is a class of models and $\Gamma(\mathcal{C})$ is the class of Γ-filtrations of models in $\mathcal{C}$, then any fórmula $\varphi \in \Gamma$ is valid in $\mathcal{C}$ if and only if it is valid in $\Gamma(\mathcal{C})$.*

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5.4 Examples of Filtrations

nml:fil:exf:sec We have not yet shown that there are any filtrations. But indeed, for any model $\mathfrak{M}$, there are many filtrations of $\mathfrak{M}$ through Γ. We identify two, in particular: the finest and coarsest filtrations. Filtrations of the same models will differ in their accessibility relation (as Definição 5.4 stipulates directly what W^* and V^* should be). The finest filtration will have as few related worlds as possible, whereas the coarsest will have as many as possible.

Definição 5.7. Where Γ is closed under subformulas, the *finest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting:

$$R^*[u][v] \quad \text{if and only if} \quad \exists u' \in [u]\ \exists v' \in [v] : Ru'v'.$$

nml:fil:exf:prop:finest **Proposição 5.8.** *The finest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Prova. We need to check that R^*, so defined, satisfies Definição 5.4(2). We check the three conditions in turn.

If Ruv then since $u \in [u]$ and $v \in [v]$, also $R^*[u][v]$, so (2a) is satisfied.

For (2b), suppose $\Box\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, u \Vdash \Box\varphi$. By definition of R^*, there are $u' \equiv u$ and $v' \equiv v$ such that $Ru'v'$. Since u and u' agree on Γ, also $\mathfrak{M}, u' \Vdash \Box\varphi$, so that $\mathfrak{M}, v' \Vdash \varphi$. By closure of Γ under sub-fórmulas, v and v' agree on φ, so $\mathfrak{M}, v \Vdash \varphi$, as desired.

To verify (2c), suppose $\Diamond\varphi \in \Gamma$, $R^*[u][v]$, and $\mathfrak{M}, v \Vdash \varphi$. By definition of R^*, there are $u' \equiv u$ and $v' \equiv v$ such that $Ru'v'$. Since v and v' agree on Γ, and Γ is closed under sub-fórmulas, also $\mathfrak{M}, v' \Vdash \varphi$, so that $\mathfrak{M}, u' \Vdash \Diamond\varphi$. Since u and u' also agree on Γ, $\mathfrak{M}, u \Vdash \Diamond\varphi$. □

Problema 5.2. Complete the proof of Proposição 5.8.

Definição 5.9. Where Γ is closed under subformulas, the *coarsest* filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ is defined by putting $R^*[u][v]$ if and only if *both* of the following conditions are met:

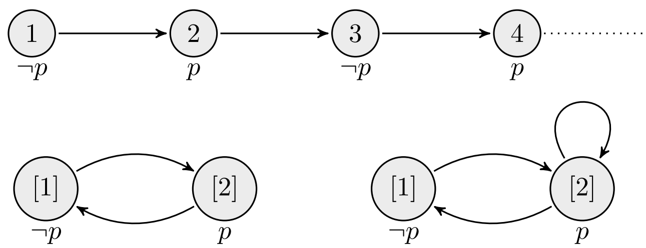


Figura 5.1: An infinite model and its filtrations.

nml:fil:exf: fig:ex-filtration
nml:fil:exf: defn:coarsest-Box
nml:fil:exf: defn:coarsest-Diamond

1. If $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Box\varphi$ then $\mathfrak{M}, v \Vdash \varphi$;
2. If $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \varphi$ then $\mathfrak{M}, u \Vdash \Diamond\varphi$.

Proposição 5.10. *The coarsest filtration $\mathfrak{M}^*$ is indeed a filtration.*

Prova. Given the definition of R^*, the only condition that is left to verify is the implication from Ruv to $R^*[u][v]$. So assume Ruv. Suppose $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Box\varphi$; then obviously $\mathfrak{M}, v \Vdash \varphi$, and (1) is satisfied. Suppose $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \varphi$. Then $\mathfrak{M}, u \Vdash \Diamond\varphi$ since Ruv, and (2) is satisfied. □

Exemplo 5.11. Let $W = \mathbb{Z}^+$, Rnm iff $m = n + 1$, and $V(p) = \{2n : n \in \mathbb{N}\}$. The model $\mathfrak{M} = \langle W, R, V\rangle$ is depicted in Figura 5.1. The worlds are 1, 2, etc.; each world can access exactly one other world—its successor—and p is true at all and only the even numbers.

Now let Γ be the set of sub-fórmulas of $\Box p \to p$, O.s., $\{p, \Box p, \Box p \to p\}$. p is true at all and only the even numbers, $\Box p$ is true at all and only the odd numbers, so $\Box p \to p$ is true at all and only the even numbers. In other words, every odd number makes $\Box p$ true but p and $\Box p \to p$ false; every even number makes p and $\Box p \to p$ true, but $\Box p$ false. So $W^* = \{[1], [2]\}$, where $[1] = \{1, 3, 5, \dots\}$ and $[2] = \{2, 4, 6, \dots\}$. Since $2 \in V(p)$, $[2] \in V^*(p)$; since $1 \notin V(p)$, $[1] \notin V^*(p)$. So $V^*(p) = \{[2]\}$.

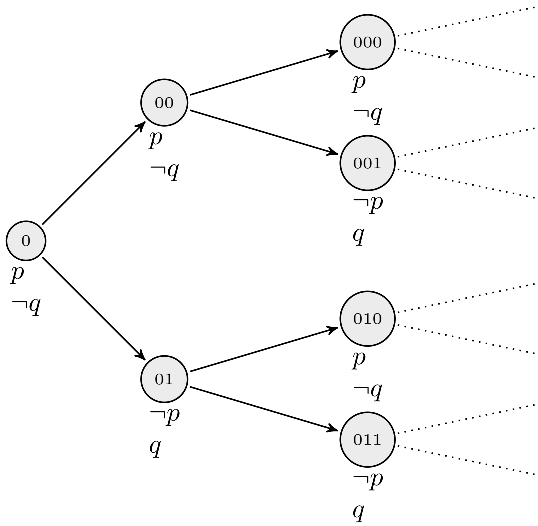
Any filtration based on W^* must have an accessibility relation that includes $\langle[1], [2]\rangle, \langle[2], [1]\rangle$: since $R12$, we must have $R^*[1][2]$ by Definição 5.4(2a), and since $R23$ we must have $R^*[2][3]$, and $[3] = [1]$. It cannot include $\langle[1], [1]\rangle$: if it did, we'd have $R^*[1][1]$, $\mathfrak{M}, 1 \Vdash \Box p$ but $\mathfrak{M}, 1 \nVdash p$, contradicting (2b). Nothing requires or rules out that $R^*[2][2]$. So, there are two possible filtrations of $\mathfrak{M}$, corresponding to the two accessibility relations

$$\{\langle[1], [2]\rangle, \langle[2], [1]\rangle\} \text{ and } \{\langle[1], [2]\rangle, \langle[2], [1]\rangle, \langle[2], [2]\rangle\}.$$

In either case, p and $\Box p \to p$ are false and $\Box p$ is true at $[1]$; p and $\Box p \to p$ are true and $\Box p$ is false at $[2]$.

Problema 5.3. Consider the following model $\mathfrak{M} = \langle W, R, V\rangle$ where $W = \{0\sigma : \sigma \in \mathbb{B}^*\}$, the set of sequences of 0s and 1s starting with 0, with $R\sigma\sigma'$ iff

$\sigma' = \sigma 0$ or $\sigma' = \sigma 1$, and $V(p) = \{\sigma 0 : \sigma \in \mathbb{B}^*\}$ and $V(q) = \{\sigma 1 : \sigma \in \mathbb{B}^* \setminus \{1\}\}$. Here's a picture:



We have $\mathfrak{M}, w \nVdash \Box(p \lor q) \to (\Box p \lor \Box q)$ for every w.

Let Γ be the set of sub-fórmulas of $\Box(p \lor q) \to (\Box p \lor \Box q)$. What are W^* and V^*? What is the accessibility relation of the finest filtration of $\mathfrak{M}$? Of the coarsest?

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5.5 Filtrations are Finite

nml:fil:fin: sec

We've defined filtrations for any set Γ that is closed under sub-fórmulas. Nothing in the definition itself guarantees that filtrations are finite. In fact, when Γ is infinite (p.ex., is the set of all fórmulas), it may well be infinite. However, if Γ is finite (p.ex., when it is the set of sub-fórmulas of a given fórmula φ), so is any filtration through Γ.

nml:fil:fin: prop:filt-are-finite

Proposição 5.12. *If Γ is finite then any filtration $\mathfrak{M}^*$ of a model $\mathfrak{M}$ through Γ is also finite.*

Prova. The size of W^* is the number of different classes $[w]$ under the equivalence relation $\equiv$. Any two worlds u, v in such class—that is, any u and v such that $u \equiv v$—agree on all fórmulas φ in Γ, $\varphi \in \Gamma$ either φ is true at both u and v, or at neither. So each class $[w]$ corresponds to subset of Γ, namely the set of all $\varphi \in \Gamma$ such that φ is true at the worlds in $[w]$. No two different classes $[u]$ and $[v]$ correspond to the same subset of Γ. For if the set of fórmulas true at u and that of fórmulas true at v are the same, then u and v agree on all formulas in Γ, O.s., $u \equiv v$. But then $[u] = [v]$. So, there is uma injetiva function from

W^* to $\wp(\Gamma)$, and hence $|W^*| \leq |\wp(\Gamma)|$. Hence if Γ contains n sentences, the cardinality of W^* is no greater than 2^n. □

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5.6 K and S5 have the Finite Model Property

Definição 5.13. A system Σ of modal logic is said to have the *finite model property* if whenever a fórmula φ is true at a world in a model of Σ then φ is true at a world in a *finite* model of Σ. nml:fil:fmp:sec

Proposição 5.14. **K** *has the finite model property.* nml:fil:fmp:prop:K-fmp

Prova. **K** is the set of valid fórmulas, O.s., any model is a model of **K**. By Teorema 5.5, if $\mathfrak{M}, w \Vdash \varphi$, then $\mathfrak{M}^*, w \Vdash \varphi$ for any filtration of $\mathfrak{M}$ through the set Γ of sub-fórmulas of φ. Any fórmula only has finitely many sub-fórmulas, so Γ is finite. By Proposição 5.12, $|W^*| \leq 2^n$, where n is the number of fórmulas in Γ. And since **K** imposes no restriction on models, $\mathfrak{M}^*$ is a **K**-model. □

To show that a logic **L** has the finite model property via filtrations it is essential that the filtration of an **L**-model is itself a **L**-model. Often this requires a fair bit of work, and not any filtration yields a **L**-model. However, for universal models, this still holds.

Proposição 5.15. *Let $\mathcal{U}$ be the class of universal models (see Proposição 2.14) and $\mathcal{U}_{\text{Fin}}$ the class of all finite universal models. Then any fórmula φ is valid in $\mathcal{U}$ if and only if it is valid in $\mathcal{U}_{\text{Fin}}$.* nml:fil:fmp:prop:univ-fin

Prova. Finite universal models are universal models, so the left-to-right direction is trivial. For the right-to left direction, suppose that φ is false at some world w in a universal model $\mathfrak{M}$. Let Γ contain φ as well as all of its subformulas; clearly Γ is finite. Take a filtration $\mathfrak{M}^*$ of $\mathfrak{M}$; then $\mathfrak{M}^*$ is finite by Proposição 5.12, and by Teorema 5.5, φ is false at $[w]$ in $\mathfrak{M}^*$. It remains to observe that $\mathfrak{M}^*$ is also universal: given u and v, by hypothesis Ruv and by Definição 5.4(2), also $R^*[u][v]$. □

Corolário 5.16. **S5** *has the finite model property.* nml:fil:fmp:cor:S5fmp

Prova. By Proposição 2.14, if φ is true at a world in some reflexive and euclidean model then it is true at a world in a universal model. By Proposição 5.15, it is true at a world in a finite universal model (namely the filtration of the model through the set of sub-fórmulas of φ). Every universal model is also reflexive and euclidean; so φ is true at a world in a finite reflexive euclidean model. □

Problema 5.4. Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

Problema 5.5. Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.

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5.7 S5 is Decidable

nml:fil:dec: sec The finite model property gives us an easy way to show that systems of modal logic given by schemas are *decidable* (O.s., that there is a computable procedure to determine whether a fórmula is derivável in the system or not).

Teorema 5.17. **S5** *is decidable.*

Prova. Let φ be given, and suppose the variáveis proposicionais occurring in φ are among $p_1, \dots, p_k$. Since for each n there are only finitely many models with n worlds assigning a value to $p_1, \dots, p_k$, we can enumerate, *in parallel*, all the theorems of **S5** by generating proofs in some systematic way; and all the models containing 1, 2, ... worlds and checking whether φ fails at a world in some such model. Eventually one of the two parallel processes will give an answer, as by Teorema 4.17 and Corolário 5.16, either φ is derivável or it fails in a finite universal model. □

The above proof works for **S5** because filtrations of universal models are automatically universal. The same holds for reflexivity and seriality, but more work is needed for other properties.
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5.8 Filtrations and Properties of Accessibility

nml:fil:acc: sec As noted, filtrations of universal, serial, and reflexive models are always also universal, serial, or reflexive. But not every filtration of a symmetric or transitive model is symmetric or transitive, respectively. In some cases, however, it is possible to define filtrations so that this does hold. In order to do so, we proceed as in the definition of the coarsest filtration, but add additional conditions to the definition of R^*. Let Γ be closed under sub-fórmulas. Consider the relations $C_i(u, v)$ in Tabela 5.1 between worlds u, v in a model $\mathfrak{M} = \langle W, R, V \rangle$. We can define $R^*[u][v]$ on the basis of combinations of these conditions. For instance, if we stipulate that $R^*[u][v]$ iff the condition $C_1(u, v)$ holds, we get exactly the coarsest filtration. If we stipulate $R^*[u][v]$ iff both $C_1(u, v)$ and $C_2(u, v)$ hold, we get a different filtration. It is "finer" than the coarsest since fewer pairs of worlds satisfy $C_1(u, v)$ and $C_2(u, v)$ than $C_1(u, v)$ alone.

nml:fil:acc: thm:more-filtrations **Teorema 5.18.** *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model, Γ closed under sub-fórmulas. Let W^* and V^* be defined as in Definição 5.4. Then:*

1. *Suppose $R^*[u][v]$ if and only if $C_1(u, v) \wedge C_2(u, v)$. Then R^* is symmetric, and $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a filtration if $\mathfrak{M}$ is symmetric.*

$C_1(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Box\varphi$ then $\mathfrak{M}, v \Vdash \varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \varphi$ then $\mathfrak{M}, u \Vdash \Diamond\varphi$;
$C_2(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \Box\varphi$ then $\mathfrak{M}, u \Vdash \varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \varphi$ then $\mathfrak{M}, v \Vdash \Diamond\varphi$;
$C_3(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Box\varphi$ then $\mathfrak{M}, v \Vdash \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \Diamond\varphi$ then $\mathfrak{M}, u \Vdash \Diamond\varphi$;
$C_4(u,v)$:	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \Box\varphi$ then $\mathfrak{M}, u \Vdash \Box\varphi$; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Diamond\varphi$ then $\mathfrak{M}, v \Vdash \Diamond\varphi$;

Tabela 5.1: Conditions on possible worlds for defining filtrations.

nml:fil:acc:
tab:Cn-filtrations

2. *Suppose $R^*[u][v]$ if and only if $C_1(u,v) \wedge C_3(u,v)$. Then R^* is transitive, and $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a filtration if $\mathfrak{M}$ is transitive.*

3. *Suppose $R^*[u][v]$ if and only if $C_1(u,v) \wedge C_2(u,v) \wedge C_3(u,v) \wedge C_4(u,v)$. Then R^* is symmetric and transitive, and $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a filtration if $\mathfrak{M}$ is symmetric and transitive.*

4. *Suppose R^* is defined as $R^*[u][v]$ if and only if $C_1(u,v) \wedge C_3(u,v) \wedge C_4(u,v)$. Then R^* is transitive and euclidean, and $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ is a filtration if $\mathfrak{M}$ is transitive and euclidean.*

Prova. 1. It's immediate that R^* is symmetric, since $C_1(u,v) \Leftrightarrow C_2(v,u)$ and $C_2(u,v) \Leftrightarrow C_1(v,u)$. So it's left to show that if $\mathfrak{M}$ is symmetric then $\mathfrak{M}^*$ is a filtration through Γ. Condition $C_1(u,v)$ guarantees that (2b) and (2c) of Definição 5.4 are satisfied. So we just have to verify Definição 5.4(2a), O.s., that Ruv implies $R^*[u][v]$.

So suppose Ruv. To show $R^*[u][v]$ we need to establish that $C_1(u,v)$ and $C_2(u,v)$. For C_1: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Box\varphi$ then also $\mathfrak{M}, v \Vdash \varphi$ (since Ruv). Similarly, if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \varphi$ then $\mathfrak{M}, u \Vdash \Diamond\varphi$ since Ruv. For C_2: if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \Box\varphi$ then Ruv implies Rvu by symmetry, so that $\mathfrak{M}, u \Vdash \varphi$. Similarly, if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \varphi$ then $\mathfrak{M}, v \Vdash \Diamond\varphi$ (since Rvu by symmetry).

2. Exercise.

3. Exercise.

4. Exercise. □

Problema 5.6. Complete the proof of Teorema 5.18.

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5.9 Filtrations of Euclidean Models

nml:fil:euc:sec

The approach of Seção 5.8 does not work in the case of models that are euclidean or serial and euclidean. Consider the model at the top of Figura 5.2, which is both euclidean and serial. Let $\Gamma = \{p, \Box p\}$. When taking a filtration through Γ, then $[w_1] = [w_3]$ since w_1 and w_3 are the only worlds that agree on Γ. Any filtration will also have the arrow inherited from $\mathfrak{M}$, as depicted in Figura 5.3. That model isn't euclidean. Moreover, we cannot add arrows to that model in order to make it euclidean. We would have to add double arrows between $[w_2]$ and $[w_4]$, and then also between w_2 and w_5. But $\Box p$ is supposed to be true at w_2, while p is false at w_5.

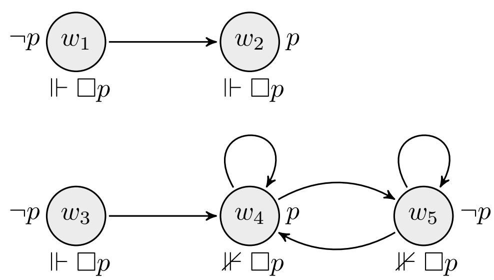

Figura 5.2: A serial and euclidean model.

nml:fil:euc:fig:ser-eucl

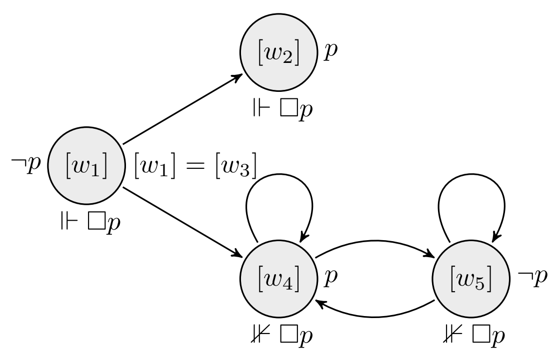

Figura 5.3: The filtration of the model in Figura 5.2.

nml:fil:euc:fig:ser-eucl2

In particular, to obtain a euclidean filtration it is not enough to consider filtrations through arbitrary Γ's closed under sub-fórmulas. Instead we need to consider sets Γ that are *modally closed* (see Definição 5.1). Such sets of sentences are infinite, and therefore do not immediately yield a finite model property or the decidability of the corresponding system.

nml:fil:euc:thm:modal-closed-filt

Teorema 5.19. *Let Γ be modally closed, $\mathfrak{M} = \langle W, R, V\rangle$, and $\mathfrak{M}^* = \langle W^*, R^*, V^*\rangle$ be a coarsest filtration of $\mathfrak{M}$.*

1. *If $\mathfrak{M}$ is symmetric, so is $\mathfrak{M}^*$.*

2. *If $\mathfrak{M}$ is transitive, so is $\mathfrak{M}^*$.*

3. *If $\mathfrak{M}$ is euclidean, so is $\mathfrak{M}^*$.*

Prova. 1. If $\mathfrak{M}^*$ is a coarsest filtration, then by definition $R^*[u][v]$ holds if and only if $C_1(u, v)$. For transitivity, suppose $C_1(u, v)$ and $C_1(v, w)$; we have to show $C_1(u, w)$. Suppose $\mathfrak{M}, u \Vdash \Box\varphi$; then $\mathfrak{M}, u \Vdash \Box\Box\varphi$ since 4 is valid in all transitive models; since $\Box\Box\varphi \in \Gamma$ by closure, also by $C_1(u, v)$, $\mathfrak{M}, v \Vdash \Box\varphi$ and by $C_1(v, w)$, also $\mathfrak{M}, w \Vdash \varphi$. Suppose $\mathfrak{M}, w \Vdash \varphi$; then $\mathfrak{M}, v \Vdash \Diamond\varphi$ by $C_1(v, w)$, since $\Diamond\varphi \in \Gamma$ by modal closure. By $C_1(u, v)$, we get $\mathfrak{M}, u \Vdash \Diamond\Diamond\varphi$ since $\Diamond\Diamond\varphi \in \Gamma$ by modal closure. Since $4_\Diamond$ is valid in all transitive models, $\mathfrak{M}, u \Vdash \Diamond\varphi$.

2. Exercise. Use the fact that both 5 and $5_\Diamond$ are valid in all euclidean models.

3. Exercise. Use the fact that B and $\mathrm{B}_\Diamond$ are valid in all symmetric models. □

Problema 5.7. Complete the proof of Teorema 5.19.

Capítulo 6

Modal Tableaux

Draft chapter on prefixed tableaux for modal logic. Needs more examples, completeness proofs, and discussion of how one can find countermodels from unsuccessful searches for closed tableaux.

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6.1 Introduction

nml:tab:int:sec Tableaux are certain (downward-branching) trees of fórmulas assinadas, O.s., pairs consisting of a truth value sign ($\mathbb{T}$ or $\mathbb{F}$) and a sentença

$$\mathbb{T}\,\varphi \text{ or } \mathbb{F}\,\varphi.$$

A tableau begins with a number of *assumptions*. Each further fórmula assinada is generated by applying one of the inference rules. Some inference rules add one or more fórmulas assinadas to a tip of the tree; others add two new tips, resulting in two branches. Rules result in fórmulas assinadas where the fórmula is less complex than that of the fórmula assinada to which it was applied. When a branch contains both $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$, we say the branch is *closed*. If every branch in a tableau is closed, the entire tableau is closed. A closed tableau constitutes a derivação that shows that the set of fórmulas assinadas which were used to begin the tableau are unsatisfiable. This can be used to define a $\vdash$ relation: $\Gamma \vdash \varphi$ iff there is some finite set $\Gamma_0 = \{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ such that there is a closed tableau for the assumptions

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

For modal logics, we have to both extend the notion of fórmula assinada and add rules that cover $\Box$ and $\Diamond$ In addition to a sign($\mathbb{T}$ or $\mathbb{F}$), fórmulas in modal tableaux also have *prefixes* σ. The prefixes are non-empty sequences of

$\dfrac{\sigma\,\mathbb{T}\,\neg\varphi}{\sigma\,\mathbb{F}\,\varphi}\;\neg\mathbb{T}$	$\dfrac{\sigma\,\mathbb{F}\,\neg\varphi}{\sigma\,\mathbb{T}\,\varphi}\;\neg\mathbb{F}$
$\dfrac{\sigma\,\mathbb{T}\,\varphi\land\psi}{\begin{array}{c}\sigma\,\mathbb{T}\,\varphi\\ \sigma\,\mathbb{T}\,\psi\end{array}}\;\land\mathbb{T}$	$\dfrac{\sigma\,\mathbb{F}\,\varphi\land\psi}{\sigma\,\mathbb{F}\,\varphi \quad\mid\quad \sigma\,\mathbb{F}\,\psi}\;\land\mathbb{F}$
$\dfrac{\sigma\,\mathbb{T}\,\varphi\lor\psi}{\sigma\,\mathbb{T}\,\varphi \quad\mid\quad \sigma\,\mathbb{T}\,\psi}\;\lor\mathbb{T}$	$\dfrac{\sigma\,\mathbb{F}\,\varphi\lor\psi}{\begin{array}{c}\sigma\,\mathbb{F}\,\varphi\\ \sigma\,\mathbb{F}\,\psi\end{array}}\;\lor\mathbb{F}$
$\dfrac{\sigma\,\mathbb{T}\,\varphi\to\psi}{\sigma\,\mathbb{F}\,\varphi \quad\mid\quad \sigma\,\mathbb{T}\,\psi}\;\to\mathbb{T}$	$\dfrac{\sigma\,\mathbb{F}\,\varphi\to\psi}{\begin{array}{c}\sigma\,\mathbb{T}\,\varphi\\ \sigma\,\mathbb{F}\,\psi\end{array}}\;\to\mathbb{F}$

Tabela 6.1: Prefixed tableau rules for the propositional connectives

nml:tab:rul:
tab:prop-rules

positive integers, O.s., $\sigma \in (\mathbb{Z}^+)^* \setminus \{\Lambda\}$. When we write such prefixes without the surrounding $\langle\ \rangle$, and separate the individual membros by .'s instead of ,'s. If σ is a prefix, then $\sigma.n$ is $\sigma \frown \langle n\rangle$; p.ex., if $\sigma = 1.2.1$, then $\sigma.3$ is $1.2.1.3$. So for instance,

$$1.2\,\mathbb{T}\,\Box\varphi \to \varphi$$

is a *prefixed fórmula assinada* (or just a *prefixed fórmula* for short).

Intuitively, the prefix names a world in a model that might satisfy the fórmulas on a branch of a tableau, and if σ names some world, then $\sigma.n$ names a world accessible from (the world named by) σ.

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6.2 Rules for K

nml:tab:rul:
sec

The rules for the regular propositional connectives are the same as for regular propositional signed tableaux, just with prefixes added. In each case, the rule applied to a signed fórmula $\sigma\,S\,\varphi$ produces new fórmulas that are also prefixed by σ. This should be intuitively clear: p.ex., if $\varphi\land\psi$ is true at (a world named by) σ, then φ and ψ are true at σ (and not at any other world). We collect the propositional rules in Tabela 6.1.

The closure condition is the same as for ordinary tableaux, although we require that not just the fórmulas but also the prefixes must match. So a branch is closed if it contains both

$$\sigma\,\mathbb{T}\,\varphi \quad\text{and}\quad \sigma\,\mathbb{F}\,\varphi$$

for some prefix σ and fórmula φ.

$\dfrac{\sigma\,\mathbb{T}\,\Box\varphi}{\sigma.n\,\mathbb{T}\,\varphi}\,\Box\mathbb{T}$ $\sigma.n$ is used	$\dfrac{\sigma\,\mathbb{F}\,\Box\varphi}{\sigma.n\,\mathbb{F}\,\varphi}\,\Box\mathbb{F}$ $\sigma.n$ is new
$\dfrac{\sigma\,\mathbb{T}\,\Diamond\varphi}{\sigma.n\,\mathbb{T}\,\varphi}\,\Diamond\mathbb{T}$ $\sigma.n$ is new	$\dfrac{\sigma\,\mathbb{F}\,\Diamond\varphi}{\sigma.n\,\mathbb{F}\,\varphi}\,\Diamond\mathbb{F}$ $\sigma.n$ is used

Tabela 6.2: The modal rules for K.

nml:tab:rul:
tab:rules-K

The rules for setting up assumptions is also as for ordinary tableaux, except that for assumptions we always use the prefix 1. (It does not matter which prefix we use, as long as it's the same for all assumptions.) So, p.ex., we say that

$$\psi_1, \dots, \psi_n \vdash \varphi$$

iff there is a closed tableau for the assumptions

$$1\,\mathbb{T}\,\psi_1, \dots, 1\,\mathbb{T}\,\psi_n, 1\,\mathbb{F}\,\varphi.$$

For the modal operators $\Box$ and $\Diamond$, the prefix of the conclusion of the rule applied to a fórmula with prefix σ is $\sigma.n$. However, which n is allowed depends on whether the sign is $\mathbb{T}$ or $\mathbb{F}$.

The $\Box\mathbb{T}$ rule extends a branch containing $\sigma\,\mathbb{T}\,\Box\varphi$ by $\sigma.n\,\mathbb{T}\,\varphi$. Similarly, the $\Diamond\mathbb{F}$ rule extends a branch containing $\sigma\,\mathbb{F}\,\Diamond\varphi$ by $\sigma.n\,\mathbb{F}\,\varphi$. They can only be applied for a prefix $\sigma.n$ which *already* occurs on the branch in which it is applied. Let's call such a prefix "used" (on the branch).

The $\Box\mathbb{F}$ rule extends a branch containing $\sigma\,\mathbb{F}\,\Box\varphi$ by $\sigma.n\,\mathbb{F}\,\varphi$. Similarly, the $\Diamond\mathbb{T}$ rule extends a branch containing $\sigma\,\mathbb{T}\,\Diamond\varphi$ by $\sigma.n\,\mathbb{T}\,\varphi$. These rules, however, can only be applied for a prefix $\sigma.n$ which *does not* already occur on the branch in which it is applied. We call such prefixes "new" (to the branch).

The rules are given in Tabela 6.2.

The requirement that the restriction that the prefix for $\Box\mathbb{T}$ must be used is necessary as otherwise we would count the following as a closed tableau:

1.	$1\,\mathbb{T}$	$\Box\varphi$	Assumption
2.	$1\,\mathbb{F}$	$\Diamond\varphi$	Assumption
3.	$1.1\,\mathbb{T}$	φ	$\Box\mathbb{T}\,1$
4.	$1.1\,\mathbb{F}$	φ	$\Diamond\mathbb{F}\,2$
	$\otimes$		

But $\Box\varphi \nvDash \Diamond\varphi$, so our proof system would be unsound. Likewise, $\Diamond\varphi \nvDash \Box\varphi$, but without the restriction that the prefix for $\Box\mathbb{F}$ must be new, this would be a closed tableau:

1.	$1\,\mathbb{T}\;\;\Diamond\varphi$	Assumption
2.	$1\,\mathbb{F}\;\;\Box\varphi$	Assumption
3.	$1.1\,\mathbb{T}\;\;\varphi$	$\Diamond\mathbb{T}\,1$
4.	$1.1\,\mathbb{F}\;\;\varphi$	$\Box\mathbb{F}\,2$
	$\otimes$	

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6.3 Tableaux for K

nml:tab:prk:sec

Exemplo 6.1. We give a closed tableau that shows $\vdash (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$.

1.	$1\,\mathbb{F}\;\;(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$		Assumption
2.	$1\,\mathbb{T}\;\;\Box\varphi \wedge \Box\psi$		$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\;\;\Box(\varphi \wedge \psi)$		$\to\mathbb{F}\,1$
4.	$1\,\mathbb{T}\;\;\Box\varphi$		$\wedge\mathbb{T}\,2$
5.	$1\,\mathbb{T}\;\;\Box\psi$		$\wedge\mathbb{T}\,2$
6.	$1.1\,\mathbb{F}\;\;\varphi \wedge \psi$		$\Box\mathbb{F}\,3$
7.	$1.1\,\mathbb{F}\;\;\varphi$	$1.1\,\mathbb{F}\;\;\psi$	$\wedge\mathbb{F}\,6$
8.	$1.1\,\mathbb{T}\;\;\varphi$	$1.1\,\mathbb{T}\;\;\psi$	$\Box\mathbb{T}\,4$; $\Box\mathbb{T}\,5$
	$\otimes$	$\otimes$	

Exemplo 6.2. We give a closed tableau that shows $\vdash \Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$:

1.	$1\,\mathbb{F}\;\;\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$		Assumption
2.	$1\,\mathbb{T}\;\;\Diamond(\varphi \vee \psi)$		$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\;\;\Diamond\varphi \vee \Diamond\psi$		$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\;\;\Diamond\varphi$		$\vee\mathbb{F}\,3$
5.	$1\,\mathbb{F}\;\;\Diamond\psi$		$\vee\mathbb{F}\,3$
6.	$1.1\,\mathbb{T}\;\;\varphi \vee \psi$		$\Diamond\mathbb{T}\,2$
7.	$1.1\,\mathbb{T}\;\;\varphi$	$1.1\,\mathbb{T}\;\;\psi$	$\vee\mathbb{T}\,6$
8.	$1.1\,\mathbb{F}\;\;\varphi$	$1.1\,\mathbb{F}\;\;\psi$	$\Diamond\mathbb{F}\,4$; $\Diamond\mathbb{F}\,5$
	$\otimes$	$\otimes$	

Problema 6.1. Find closed tableaux in **K** for the following fórmulas:

1. $\Box\neg p \to \Box(p \to q)$

2. $(\Box p \lor \Box q) \to \Box(p \lor q)$

3. $\Diamond p \to \Diamond(p \lor q)$

4. $\Box(p \land q) \to \Box p$

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6.4 Soundness for K

nml:tab:sou:
sec

> This soundness proof reuses the soundness proof for classical propositional logic, O.s., it proves everything from scratch. That's ok if you want a self-contained soundness proof. If you already have seen soundness for ordinary tableau this will be repetitive. It's planned to make it possible to switch between self-contained version and a version building on the non-modal case.

In order to show that prefixed tableaux are sound, we have to show that if explanation

$$1\,\mathbb{T}\,\psi_1, \dots, 1\,\mathbb{T}\,\psi_n, 1\,\mathbb{F}\,\varphi$$

has a closed tableau then $\psi_1, \dots, \psi_n \vDash \varphi$. It is easier to prove the contrapositive: if for some $\mathfrak{M}$ and world w, $\mathfrak{M}, w \Vdash \psi_i$ for all $i = 1, \dots, n$ but $\mathfrak{M}, w \Vdash \varphi$, then no tableau can close. Such a countermodel shows that the initial assumptions of the tableau are satisfiable. The strategy of the proof is to show that whenever all the prefixed fórmulas on a tableau branch are satisfiable, any application of a rule results in at least one extended branch that is also satisfiable. Since closed branches are unsatisfiable, any tableau for a satisfiable set of prefixed fórmulas must have at least one open branch.

In order to apply this strategy in the modal case, we have to extend our definition of "satisfiable" to modal modals and prefixes. With that in hand, however, the proof is straightforward.

Definição 6.3. Let P be some set of prefixes, O.s., $P \subseteq (\mathbb{Z}^+)^* \setminus \{\Lambda\}$ and let $\mathfrak{M}$ be a model. A function $f \colon P \to W$ is an *interpretation of* P in $\mathfrak{M}$ if, whenever σ and $\sigma.n$ are both in P, then $Rf(\sigma)f(\sigma.n)$.

Relative to an interpretation of prefixes P we can define:

1. $\mathfrak{M}$ satisfies $\sigma\,\mathbb{T}\,\varphi$ iff $\mathfrak{M}, f(\sigma) \Vdash \varphi$.

2. $\mathfrak{M}$ satisfies $\sigma\,\mathbb{F}\,\varphi$ iff $\mathfrak{M}, f(\sigma) \nVdash \varphi$.

Definição 6.4. Let Γ be a set of prefixed fórmulas, and let $P(\Gamma)$ be the set of prefixes that occur in it. If f is an interpretation of $P(\Gamma)$ in $\mathfrak{M}$, we say that $\mathfrak{M}$ satisfies Γ with respect to f, $\mathfrak{M}, f \Vdash \Gamma$, if $\mathfrak{M}$ satisfies every prefixed fórmula in Γ with respect to f. Γ is *satisfiable* iff there is a model $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that $\mathfrak{M}, f \Vdash \Gamma$.

Proposição 6.5. *If Γ contains both $\sigma\,\mathbb{T}\,\varphi$ and $\sigma\,\mathbb{F}\,\varphi$, for some fórmula φ and prefix σ, then Γ is unsatisfiable.*

Prova. There cannot be a model $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that both $\mathfrak{M}, f(\sigma) \Vdash \varphi$ and $\mathfrak{M}, f(\sigma) \nVdash \varphi$. □

Teorema 6.6 (Soundness). *If Γ has a closed tableau, Γ is unsatisfiable.* *nml:tab:sou: thm:tableau-soundness*

Prova. We call a branch of a tableau satisfiable iff the set of fórmulas assinadas on it is satisfiable, and let's call a tableau satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable tableau by one of the rules of inference always results in a satisfiable tableau. This will prove the theorem: any closed tableau results by applying rules of inference to the tableau consisting only of assumptions from Γ. So if Γ were satisfiable, any tableau for it would be satisfiable. A closed tableau, however, is clearly not satisfiable, since all its branches are closed and closed branches are unsatisfiable.

Suppose we have a satisfiable tableau, O.s., a tableau with at least one satisfiable branch. Applying a rule of inference either adds fórmulas assinadas to a branch, or splits a branch in two. If the tableau has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended tableau, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let Γ be the set of fórmulas assinadas on that branch, and let $\sigma\,S\,\varphi \in \Gamma$ be the fórmula assinada to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, O.s., Γ together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable. First, we consider the possible inferences with only one premise.

1. The branch is expanded by applying $\neg\mathbb{T}$ to $\sigma\,\mathbb{T}\,\neg\psi \in \Gamma$. Then the extended branch contains the fórmulas assinadas $\Gamma \cup \{\sigma\,\mathbb{F}\,\psi\}$. Suppose $\mathfrak{M}, f \Vdash \Gamma$. In particular, $\mathfrak{M}, f(\sigma) \Vdash \neg\psi$. Thus, $\mathfrak{M}, f(\sigma) \nVdash \psi$, O.s., $\mathfrak{M}$ satisfies $\sigma\,\mathbb{F}\,\psi$ with respect to f.

2. The branch is expanded by applying $\neg\mathbb{F}$ to $\sigma\,\mathbb{F}\,\neg\psi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\wedge\mathbb{T}$ to $\sigma\,\mathbb{T}\,\psi \wedge \chi \in \Gamma$, which results in two new fórmulas assinadas on the branch: $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \Vdash \psi \wedge \chi$. Then $\mathfrak{M}, f(\sigma) \Vdash \psi$ and $\mathfrak{M}, f(\sigma) \Vdash \chi$. This means that $\mathfrak{M}$ satisfies both $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$ with respect to f.

4. The branch is expanded by applying $\vee\mathbb{F}$ to $\mathbb{F}\,\psi \vee \chi \in \Gamma$: Exercise.

5. The branch is expanded by applying $\to\mathbb{F}$ to $\sigma\,\mathbb{F}\,\psi \to \chi \in \Gamma$: This results in two new fórmulas assinadas on the branch: $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{F}\,\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \nVdash \psi \to \chi$. Then $\mathfrak{M}, f(\sigma) \Vdash \psi$ and $\mathfrak{M}, f(\sigma) \nVdash \chi$. This means that $\mathfrak{M}, f$ satisfies both $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{F}\,\chi$.

6. The branch is expanded by applying $\Box\mathbb{T}$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new fórmula assinada $\sigma.n\,\mathbb{T}\,\psi$ on the branch, for some $\sigma.n \in P(\Gamma)$ (since $\sigma.n$ must be used). Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since f is an interpretation of prefixes and both σ, $\sigma.n \in P(\Gamma)$, we know that $Rf(\sigma)f(\sigma.n)$. Hence, $\mathfrak{M}, f(\sigma.n) \Vdash \psi$, O.s., $\mathfrak{M}, f$ satisfies $\sigma.n\,\mathbb{T}\,\psi$.

7. The branch is expanded by applying $\Box\mathbb{F}$ to $\sigma\,\mathbb{F}\,\Box\psi \in \Gamma$: This results in a new fórmula assinada $\sigma.n\,\mathbb{F}\,\varphi$, where $\sigma.n$ is a new prefix on the branch, O.s., $\sigma.n \notin P(\Gamma)$. Since Γ is satisfiable, there is a $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that $\mathfrak{M}, f \vDash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \nVdash \Box\psi$. We have to show that $\Gamma \cup \{\sigma.n\,\mathbb{F}\,\psi\}$ is satisfiable. To do this, we define an interpretation of $P(\Gamma) \cup \{\sigma.n\}$ as follows:

   Since $\mathfrak{M}, f(\sigma) \nVdash \Box\psi$, there is a $w \in W$ such that $Rf(\sigma)w$ and $\mathfrak{M}, w \nVdash \psi$. Let f' be like f, except that $f'(\sigma.n) = w$. Since $f'(\sigma) = f(\sigma)$ and $Rf(\sigma)w$, we have $Rf'(\sigma)f'(\sigma.n)$, so f' is an interpretation of $P(\Gamma) \cup \{\sigma.n\}$. Obviously $\mathfrak{M}, f'(\sigma.n) \nVdash \psi$. Since $f(\sigma') = f'(\sigma')$ for all prefixes $\sigma' \in P(\Gamma)$, $\mathfrak{M}, f' \Vdash \Gamma$. So, $\mathfrak{M}, f'$ satisfies $\Gamma \cup \{\sigma.n\,\mathbb{F}\,\psi\}$.

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying $\wedge\mathbb{F}$ to $\sigma\,\mathbb{F}\,\psi \wedge \chi \in \Gamma$, which results in two branches, a left one continuing through $\sigma\,\mathbb{F}\,\psi$ and a right one through $\sigma\,\mathbb{F}\,\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \nVdash \psi \wedge \chi$. Then $\mathfrak{M}, f(\sigma) \nVdash \psi$ or $\mathfrak{M}, f(\sigma) \nVdash \chi$. In the former case, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\psi$, O.s., the left branch is satisfiable. In the latter, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\chi$, O.s., the right branch is satisfiable.

2. The branch is expanded by applying $\vee\mathbb{T}$ to $\sigma\,\mathbb{T}\,\psi \vee \chi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\rightarrow\mathbb{T}$ to $\sigma\,\mathbb{T}\,\psi \rightarrow \chi \in \Gamma$: Exercise. □

Problema 6.2. Complete the proof of Teorema 6.6.

nml:tab:sou: cor:entailment-soundness

Corolário 6.7. *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

Prova. If $\Gamma \vdash \varphi$ then for some $\psi_1, \ldots, \psi_n \in \Gamma$, $\Delta = \{1\,\mathbb{F}\,\varphi, 1\,\mathbb{T}\,\psi_1, \ldots, 1\,\mathbb{T}\,\psi_n\}$ has a closed tableau. We want to show that $\Gamma \vDash \varphi$. Suppose not, so for some $\mathfrak{M}$ and w, $\mathfrak{M}, w \Vdash \psi_i$ for $i = 1, \ldots, n$, but $\mathfrak{M}, w \nVdash \varphi$. Let $f(1) = w$; then f is an interpretation of $P(\Delta)$ into $\mathfrak{M}$, and $\mathfrak{M}$ satisfies Δ with respect to f. But by Teorema 6.6, Δ is unsatisfiable since it has a closed tableau, a contradiction. So we must have $\Gamma \vdash \varphi$ after all. □

nml:tab:sou: cor:weak-soundness

Corolário 6.8. *If $\vdash \varphi$ then φ is true in all models.*

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$\dfrac{\sigma\, \mathbb{T}\, \Box\varphi}{\sigma\, \mathbb{T}\, \varphi}$ T$\Box$	$\dfrac{\sigma\, \mathbb{F}\, \Diamond\varphi}{\sigma\, \mathbb{F}\, \varphi}$ T$\Diamond$
$\dfrac{\sigma\, \mathbb{T}\, \Box\varphi}{\sigma\, \mathbb{T}\, \Diamond\varphi}$ D$\Box$	$\dfrac{\sigma\, \mathbb{F}\, \Diamond\varphi}{\sigma\, \mathbb{F}\, \Box\varphi}$ D$\Diamond$
$\dfrac{\sigma.n\, \mathbb{T}\, \Box\varphi}{\sigma\, \mathbb{T}\, \varphi}$ B$\Box$	$\dfrac{\sigma.n\, \mathbb{F}\, \Diamond\varphi}{\sigma\, \mathbb{F}\, \varphi}$ B$\Diamond$
$\dfrac{\sigma\, \mathbb{T}\, \Box\varphi}{\sigma.n\, \mathbb{T}\, \Box\varphi}$ 4$\Box$ $\sigma.n$ is used	$\dfrac{\sigma\, \mathbb{F}\, \Diamond\varphi}{\sigma.n\, \mathbb{F}\, \Diamond\varphi}$ 4$\Diamond$ $\sigma.n$ is used
$\dfrac{\sigma.n\, \mathbb{T}\, \Box\varphi}{\sigma\, \mathbb{T}\, \Box\varphi}$ 4r$\Box$	$\dfrac{\sigma.n\, \mathbb{F}\, \Diamond\varphi}{\sigma\, \mathbb{F}\, \Diamond\varphi}$ 4r$\Diamond$

Tabela 6.3: More modal rules.

nml:tab:mru: tab:more-rules

Logic	R is ...	Rules
T = **KT**	reflexive	T$\Box$, T$\Diamond$
D = **KD**	serial	D$\Box$, D$\Diamond$
K4	transitive	4$\Box$, 4$\Diamond$
B = **KTB**	reflexive, symmetric	T$\Box$, T$\Diamond$ B$\Box$, B$\Diamond$
S4 = **KT4**	reflexive, transitive	T$\Box$, T$\Diamond$, 4$\Box$, 4$\Diamond$
S5 = **KT4B**	reflexive, transitive, euclidean	T$\Box$, T$\Diamond$, 4$\Box$, 4$\Diamond$, 4r$\Box$, 4r$\Diamond$

Tabela 6.4: Tableau rules for various modal logics.

nml:tab:mru: tab:logics-rules

6.5 Rules for Other Accessibility Relations

nml:tab:mru: sec

In order to deal with logics determined by special accessibility relations, we consider the additional rules in Tabela 6.3.

Adding these rules results in systems that are sound and complete for the logics given in Tabela 6.4.

Exemplo 6.9. We give a closed tableau that shows $\mathbf{S5} \vdash 5$, O.s., $\Box\varphi \to \Box\Diamond\varphi$.

1.	$1\,\mathbb{F}\ \ \Box\varphi \to \Box\Diamond\varphi$	Assumption
2.	$1\,\mathbb{T}\ \ \Box\varphi$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \ \Box\Diamond\varphi$	$\to\mathbb{F}\,1$
4.	$1.1\,\mathbb{F}\ \ \Diamond\varphi$	$\Box\mathbb{F}\,3$
5.	$1\,\mathbb{F}\ \ \Diamond\varphi$	$4\mathrm{r}\Diamond\ 4$
6.	$1.1\,\mathbb{F}\ \ \varphi$	$\Diamond\mathbb{F}\,5$
7.	$1.1\,\mathbb{T}\ \ \varphi$	$\Box\mathbb{T}\,2$
	$\otimes$	

Problema 6.3. Give closed tableaux that show the following:

1. $\mathbf{KT5} \vdash \mathrm{B}$;
2. $\mathbf{KT5} \vdash 4$;
3. $\mathbf{KDB4} \vdash \mathrm{T}$;
4. $\mathbf{KB4} \vdash 5$;
5. $\mathbf{KB5} \vdash 4$;
6. $\mathbf{KT} \vdash \mathrm{D}$.

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6.6 Soundness for Additional Rules

nml:tab:msn:sec We say a rule is sound for a class of models if, whenever a branch in a tableau is satisfiable in a model from that class, the branch resulting from applying the rule is also satisfiable in a model from that class.

nml:tab:msn:prop:soundness-T **Proposição 6.10.** $\mathrm{T}\Box$ *and* $\mathrm{T}\Diamond$ *are sound for reflexive models.*

Prova.

1. The branch is expanded by applying $\mathrm{T}\Box$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new fórmula assinada $\sigma\,\mathbb{T}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \psi$, O.s., $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\psi$.

2. The branch is expanded by applying $\mathrm{T}\Diamond$ to $\sigma\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new fórmula assinada $\sigma\,\mathbb{F}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \nVdash \psi$, O.s., $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\psi$. □

nml:tab:msn:prop:soundness-D **Proposição 6.11.** $\mathrm{D}\Box$ *and* $\mathrm{D}\Diamond$ *are sound for serial models.*

Prova. 1. The branch is expanded by applying D$\Box$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new fórmula assinada $\sigma\,\mathbb{T}\,\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \Vdash \psi$, and hence $\mathfrak{M}, f(\sigma) \Vdash \Diamond\psi$. So, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\Diamond\psi$.

2. The branch is expanded by applying D$\Diamond$ to $\sigma\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new fórmula assinada $\sigma\,\mathbb{F}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \nVdash \psi$, and hence $\mathfrak{M}, f(\sigma) \nVdash \Box\psi$. So, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\Box\psi$. □

Proposição 6.12. B$\Box$ *and* B$\Diamond$ *are sound for symmetric models.* *nml:tab:msn:prop:soundness-B*

Prova. 1. The branch is expanded by applying B$\Box$ to $\sigma.n\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new fórmula assinada $\sigma\,\mathbb{T}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, $\mathfrak{M}, f(\sigma) \Vdash \psi$. Hence, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\psi$.

2. The branch is expanded by applying B$\Diamond$ to $\sigma.n\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new fórmula assinada $\sigma\,\mathbb{F}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$, $\mathfrak{M}, f(\sigma) \nVdash \psi$. Hence, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\psi$. □

Proposição 6.13. 4$\Box$ *and* 4$\Diamond$ *are sound for transitive models.* *nml:tab:msn:prop:soundness-4*

Prova. 1. The branch is expanded by applying 4$\Box$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new fórmula assinada $\sigma.n\,\mathbb{T}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$ and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, and $\mathfrak{M}, f$ satisfies $\sigma.n\,\mathbb{T}\,\Box\psi$.

2. The branch is expanded by applying 4$\Diamond$ to $\sigma\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new fórmula assinada $\sigma.n\,\mathbb{F}\,\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$ and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$, $\mathfrak{M}, w \nVdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$, and $\mathfrak{M}, f$ satisfies $\sigma.n\,\mathbb{F}\,\Diamond\psi$. □

nml:tab:msn: prop:soundness-4r **Proposição 6.14.** *4r$\Box$ and 4r$\Diamond$ are sound for euclidean models.*

Prova. 1. The branch is expanded by applying 4r$\Box$ to $\sigma.n\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new fórmula assinada $\sigma\,\mathbb{T}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma).n \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, and $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\Box\psi$.

2. The branch is expanded by applying 4r$\Diamond$ to $\sigma.n\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new fórmula assinada $\sigma\,\mathbb{T}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma).n \nVdash \Diamond\psi$, $\mathfrak{M}, w \nVdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$, and $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\Diamond\psi$. □

nml:tab:msn: cor:soundness-logics **Corolário 6.15.** *The tableau systems given in Tabela 6.4 are sound for the respective classes of models.*

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6.7 Simple Tableaux for S5

nml:tab:s5: sec **S5** is sound and complete with respect to the class of universal models, O.s., models where every world is accessible from every world. In universal models the accessibility relation doesn't matter: "there is a world w where $\mathfrak{M}, w \Vdash \varphi$" is true if and only if there is such a w that's accessible from u. So in **S5**, we can define models as simply a set of worlds and a valuation V. This suggests that we should be able to simplify the tableau rules as well. In the general case, we take as prefixes sequences of positive integers, so that we can keep track of which such prefixes name worlds which are accessible from others: $\sigma.n$ names a world accessible from σ. But in **S5** any world is accessible from any world, so there is no need to so keep track. Instead, we can use positive integers as prefixes. The simplified rules are given in Tabela 6.5.

Exemplo 6.16. We give a simplified closed tableau that shows $\mathbf{S5} \vdash 5$, O.s., $\Diamond\varphi \to \Box\Diamond\varphi$.

1.	$1\,\mathbb{F}\;\; \Diamond\varphi \to \Box\Diamond\varphi$	Assumption
2.	$1\,\mathbb{T}\;\; \Diamond\varphi$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\;\; \Box\Diamond\varphi$	$\to\mathbb{F}\,1$
4.	$2\,\mathbb{F}\;\; \Diamond\varphi$	$\Box\mathbb{F}\,3$
5.	$3\,\mathbb{T}\;\; \varphi$	$\Diamond\mathbb{T}\,2$
6.	$3\,\mathbb{F}\;\; \varphi$	$\Diamond\mathbb{F}\,4$
	$\otimes$	

$\dfrac{n\,\mathbb{T}\,\Box\varphi}{m\,\mathbb{T}\,\varphi}\,\Box\mathbb{T}$ m is used	$\dfrac{n\,\mathbb{F}\,\Box\varphi}{m\,\mathbb{F}\,\varphi}\,\Box\mathbb{F}$ m is new
$\dfrac{n\,\mathbb{T}\,\Diamond\varphi}{m\,\mathbb{T}\,\varphi}\,\Diamond\mathbb{T}$ m is new	$\dfrac{n\,\mathbb{F}\,\Diamond\varphi}{m\,\mathbb{F}\,\varphi}\,\Diamond\mathbb{F}$ m is used

Tabela 6.5: Simplified rules for **S5**.

nml:tab:s5: tab:rules-S5

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6.8 Completeness for K

explanation To show that the method of tableaux is complete, we have to show that whenever there is no closed tableau to show $\Gamma \vdash \varphi$, then $\Gamma \nvDash \varphi$, O.s., there is a countermodel. But "there is no closed tableau" means that every way we could try to construct one has to fail to close. The trick is to see that if every such way fails to close, then a specific, *systematic and exhaustive* way also fails to close. And this systematic and exhaustive way would close if a closed tableau exists. The single tableau will contain, among its open branches, all the information required to define a countermodel. The countermodel given by an open branch in this tableau will contain the all the prefixes used on that branch as the worlds, and a variável proposicional p is true at σ iff $\sigma\,\mathbb{T}\,p$ occurs on the branch. nml:tab:cpl: sec

Definição 6.17. A branch in a tableau is called complete if, whenever it contains a prefixed fórmula $\sigma\,S\,\varphi$ to which a rule can be applied, it also contains

1. the prefixed fórmulas that are the corresponding conclusions of the rule, in the case of propositional stacking rules;

2. one of the corresponding conclusion fórmulas in the case of propositional branching rules;

3. at least one possible conclusion in the case of modal rules that require a new prefix;

4. the corresponding conclusion for every prefix occurring on the branch in the case of modal rules that require a used prefix.

For instance, a complete branch contains $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$ whenever it contains $\mathbb{T}\,\psi \land \chi$. If it contains $\sigma\,\mathbb{T}\,\psi \lor \chi$ it contains at least one of $\sigma\,\mathbb{F}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$. If it contains $\sigma\,\mathbb{F}\,\Box$ it also contains $\sigma.n\,\mathbb{F}\,\Box$ for at least one n. And whenever it contains $\sigma\,\mathbb{T}\,\Box$ it also contains $\sigma.n\,\mathbb{T}\,\Box$ for every n such that $\sigma.n$ is used on the branch. explanation

nml:tab:cpl: prop:complete-tableau

Proposição 6.18. *Every finite Γ has a tableau in which every branch is complete.*

Prova. Consider an open branch in a tableau for Γ. There are finitely many prefixed fórmulas in the branch to which a rule could be applied. In some fixed order (say, top to bottom), for each of these prefixed fórmulas for which the conditions (1)–(4) do not already hold, apply the rules that can be applied to it to extend the branch. In some cases this will result in branching; apply the rule at the tip of each resulting branch for all remaining prefixed fórmulas. Since the number of prefixed fórmulas is finite, and the number of used prefixes on the branch is finite, this procedure eventually results in (possibly many) branches extending the original branch. Apply the procedure to each, and repeat. But by construction, every branch is closed. □

nml:tab:cpl: thm:tableau-completeness

Teorema 6.19 (Completeness). *If Γ has no closed tableau, Γ is satisfiable.*

Prova. By the proposition, Γ has a tableau in which every branch is complete. Since it has no closed tableau, it thas has a tableau in which at least one branch is open and complete. Let Δ be the set of prefixed fórmulas on the branch, and $P(\Delta)$ the set of prefixes occurring in it.

We define a model $\mathfrak{M}(\Delta) = \langle P(\Delta), R, V\rangle$ where the worlds are the prefixes occurring in Δ, the accessibility relation is given by:

$$R\sigma\sigma' \quad \text{iff} \quad \sigma' = \sigma.n \quad \text{for some } n$$

and

$$V(p) = \{\sigma : \sigma\,\mathbb{T}\,p \in \Delta\}.$$

We show by induction on φ that if $\sigma\,\mathbb{T}\,\varphi \in \Delta$ then $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$, and if $\sigma\,\mathbb{F}\,\varphi \in \Delta$ then $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

1. $\varphi \equiv p$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$ then $\sigma \in V(p)$ (by definition of V) and so $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$ then $\sigma\,\mathbb{T}\,\varphi \notin \Delta$, since the branch would otherwise be closed. So $\sigma \notin V(p)$ and thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

2. $\varphi \equiv \neg\psi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then $\sigma\,\mathbb{F}\,\psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ and thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then $\sigma\,\mathbb{T}\,\psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

3. $\varphi \equiv \psi \wedge \chi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then both $\sigma \mathbb{T} \psi \in \Delta$ and $\sigma \mathbb{T} \chi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma \mathbb{F} \varphi \in \Delta$, then either $\sigma \mathbb{F} \psi \in \Delta$ or $\sigma \mathbb{F} \chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

4. $\varphi \equiv \psi \vee \chi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then either $\sigma \mathbb{T} \psi \in \Delta$ or $\sigma \mathbb{T} \chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma \mathbb{F} \varphi \in \Delta$, then both $\sigma \mathbb{F} \psi \in \Delta$ and $\sigma \mathbb{F} \chi \in \Delta$ since the branch is complete. By induction hypothesis, both $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ and $\mathfrak{M}(\Delta), \sigma \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

5. $\varphi \equiv \psi \rightarrow \chi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then either $\sigma \mathbb{F} \psi \in \Delta$ or $\sigma \mathbb{T} \chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma \mathbb{F} \varphi \in \Delta$, then both $\sigma \mathbb{T} \psi \in \Delta$ and $\sigma \mathbb{F} \chi \in \Delta$ since the branch is complete. By induction hypothesis, both $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

6. $\varphi \equiv \Box\psi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then, since the branch is complete, $\sigma.n \mathbb{T} \psi \in \Delta$ for every $\sigma.n$ used on the branch, O.s., for every $\sigma' \in P(\Delta)$ such that $R\sigma\sigma'$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma' \Vdash \psi$ for every σ' such that $R\sigma\sigma'$. Therefore, $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma \mathbb{F} \varphi \in \Delta$, then for some $\sigma.n$, $\sigma.n \mathbb{F} \psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma.n \nVdash \psi$. Since $R\sigma(\sigma.n)$, there is a σ' such that $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

7. $\varphi \equiv \Diamond\psi$: If $\sigma \mathbb{T} \varphi \in \Delta$, then for some $\sigma.n$, $\sigma.n \mathbb{T} \psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma.n \Vdash \psi$. Since $R\sigma(\sigma.n)$, there is a σ' such that $\mathfrak{M}(\Delta), \sigma' \Vdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma \mathbb{F} \varphi \in \Delta$, then, since the branch is complete, $\sigma.n \mathbb{F} \psi \in \Delta$ for every $\sigma.n$ used on the branch, O.s., for every $\sigma' \in P(\Delta)$ such that $R\sigma\sigma'$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$ for every σ' such that $R\sigma\sigma'$. Therefore, $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

Since $\Gamma \subseteq \Delta$, $\mathfrak{M}(\Delta) \Vdash \Gamma$. □

Problema 6.4. Complete the proof of Teorema 6.19.

Corolário 6.20. *If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.* nml:tab:cpl: cor:entailment-completeness

Corolário 6.21. *If φ is true in all models, then $\vdash \varphi$.* nml:tab:cpl: cor:weak-completeness

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6.9 Countermodels from Tableaux

nml:tab:cou:sec The proof of the completeness theorem doesn't just show that if $\vDash \varphi$ then $\vdash \varphi$, explanation it also gives us a method for constructing countermodels to φ if $\nvDash A$. In the case of **K**, this method constitutes a *decision procedure*. For suppose $\nvDash \varphi$. Then the proof of Proposição 6.18 gives a method for constructing a complete tableau. The method in fact always terminates. The propositional rules for **K** only add prefixed fórmulas of lower complexity, O.s., each propositional rule need only be applied once on a branch for any signed formula $\sigma\, S\, \varphi$. New prefixes are only generated by the $\Box\mathbb{F}$ and $\Diamond\mathbb{T}$ rules, and also only have to be applied once (and produce a single new prefix). $\Box\mathbb{T}$ and $\Diamond\mathbb{F}$ have to be applied potentially multiple times, but only once per prefix, and only finitely many new prefixes are generated. So the construction either results in a closed branch or a complete branch after finitely many stages.

Once a tableau with an open complete branch is constructed, the proof of Teorema 6.19 gives us an explict model that satisfies the original set of prefixed fórmulas. So not only is it the case that if $\Gamma \vDash \varphi$, then a closed tableau exists and $\Gamma \vdash \varphi$, if we look for the closed tableau in the right way and end up with a "complete" tableau, we'll not only know that $\Gamma \nvDash \varphi$ but actually be able to construct a countermodel.

Exemplo 6.22. We know that $\nvdash \Box(p \lor q) \to (\Box p \lor \Box q)$. The construction of a tableau begins with:

1.	$1\,\mathbb{F}\ \ \Box(p \lor q) \to (\Box p \lor \Box q)$ ✓	Assumption
2.	$1\,\mathbb{T}\ \ \Box(p \lor q)$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \ \Box p \lor \Box q$ ✓	$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\ \ \Box p$ ✓	$\lor\mathbb{F}\,3$
5.	$1\,\mathbb{F}\ \ \Box q$ ✓	$\lor\mathbb{F}\,3$
6.	$1.1\,\mathbb{F}\ \ p$ ✓	$\Box\mathbb{F}\,4$
7.	$1.2\,\mathbb{F}\ \ q$ ✓	$\Box\mathbb{F}\,5$

The tableau is of course not finished yet. In the next step, we consider the only line without a checkmark: the prefixed fórmula $1\,\mathbb{T}\,\Box(p \lor q)$ on line 2. The construction of the closed tableau says to apply the $\Box\mathbb{T}$ rule for every prefix used on the branch, O.s., for both 1.1 and 1.2:

1.	$1\,\mathbb{F}\ \ \Box(p \lor q) \to (\Box p \lor \Box q)$ ✓	Assumption
2.	$1\,\mathbb{T}\ \ \Box(p \lor q)$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \ \Box p \lor \Box q$ ✓	$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\ \ \Box p$ ✓	$\lor\mathbb{F}\,3$
5.	$1\,\mathbb{F}\ \ \Box q$ ✓	$\lor\mathbb{F}\,3$
6.	$1.1\,\mathbb{F}\ \ p$ ✓	$\Box\mathbb{F}\,4$
7.	$1.2\,\mathbb{F}\ \ q$ ✓	$\Box\mathbb{F}\,5$
8.	$1.1\,\mathbb{T}\ \ p \lor q$	$\Box\mathbb{T}\,2$
9.	$1.2\,\mathbb{T}\ \ p \lor q$	$\Box\mathbb{T}\,2$

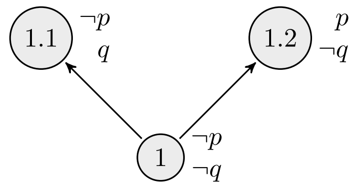

Figura 6.1: A countermodel to $\Box(p \lor q) \to (\Box p \lor \Box q)$.

nml:tab:cou: fig:counter-Box

Now lines 2, 8, and 9, don't have checkmarks. But no new prefix has been added, so we apply $\lor\mathbb{T}$ to lines 8 and 9, on all resulting branches (as long as they don't close):

1.	$1\,\mathbb{F}\ \ \Box(p \lor q) \to (\Box p \lor \Box q)$ ✓		Assumption
2.	$1\,\mathbb{T}\ \ \Box(p \lor q)$ ✓		$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \ \Box p \lor \Box q$ ✓		$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\ \ \Box p$ ✓		$\lor\mathbb{F}\,3$
5.	$1\,\mathbb{F}\ \ \Box q$ ✓		$\lor\mathbb{F}\,3$
6.	$1.1\,\mathbb{F}\ \ p$ ✓		$\Box\mathbb{F}\,4$
7.	$1.2\,\mathbb{F}\ \ q$ ✓		$\Box\mathbb{F}\,5$
8.	$1.1\,\mathbb{T}\ \ p \lor q$ ✓		$\Box\mathbb{T}\,2$
9.	$1.2\,\mathbb{T}\ \ p \lor q$ ✓		$\Box\mathbb{T}\,2$
10.	$1.1\,\mathbb{T}\ \ p$ ✓ $\otimes$	$1.1\,\mathbb{T}\ \ q$ ✓	$\lor\mathbb{T}\,8$
11.	$1.2\,\mathbb{T}\ \ p$ ✓	$1.2\,\mathbb{T}\ \ q$ ✓ $\otimes$	$\lor\mathbb{T}\,9$

There is one remaining open branch, and it is complete. From it we define the model with worlds $W = \{1, 1.1, 1.2\}$ (the only prefixes appearing on the open branch), the accessibility relation $R = \{\langle 1, 1.1\rangle, \langle 1, 1.2\rangle\}$, and the assignment $V(p) = \{1.2\}$ (because line 11 contains $1.2\,\mathbb{T}\,p$) and $V(q) = \{1.1\}$ (because line 10 contains $1.1\,\mathbb{T}\,q$). The model is pictured in Figura 6.1, and you can verify that it is a countermodel to $\Box(p \lor q) \to (\Box p \lor \Box q)$.

Capítulo 7

Modal Sequent Calculus

Draft chapter on sequent calculi for modal logic. Needs more examples, soundness and completeness proofs.

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7.1 Introduction

nml:seq:int:sec The sequent calculus for propositional logic can be extended by additional rules that deal with $\Box$ and $\Diamond$. For instance, for **K**, we have **LK** plus:

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\varphi}\,\Box \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Diamond\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}\,\Diamond$$

For extensions of **K**, additional rules have to be added as well.

Not every modal logic has such a sequent calculus. Even **S5**, which is semantically simple (it can be defined without using accessibility relations at all) is not known to have a sequent calculus that results from **LK** which is complete without the rule Cut. However, it has a cut-free complete *hypersequent* calculus.

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7.2 Rules for K

nml:seq:rul:sec The rules for the regular propositional connectives are the same as for regular sequent calculus **LK**. Axioms are also the same: any sequent of the form $\varphi \Rightarrow \varphi$ counts as an axiom.

For the modal operators $\Box$ and $\Diamond$, we have the following additional rules:

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\varphi}\,\Box \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Diamond\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}\,\Diamond$$

Here, $\Box\Gamma$ means the sequence of fórmulas resulting from Γ by putting $\Box$ in front of every fórmula in Γ and $\Diamond\Delta$ is the sequence of fórmulas resulting from Δ by putting $\Diamond$ in front of every fórmula in Δ. Γ and Δ may be empty; in that case the corresponding part $\Box\Gamma$ and $\Diamond\Delta$ of the conclusion sequent is empty as well.

The restriction of adding a $\Box$ on the right and $\Diamond$ on the left to a single fórmula φ is necessary. If we allowed to add $\Box$ to any number of fórmulas on the right or to add $\Diamond$ to any number of fórmulas on the left we would be able to deriva:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\,\neg\mathrm{R}}{\Rightarrow \Box\varphi, \Box\neg\varphi}\,\Box*}{\Rightarrow \Box\varphi \lor \Box\neg\varphi}\,\lor\mathrm{R} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg\mathrm{L}}{\Diamond\neg\varphi, \Diamond\varphi \Rightarrow}\,\Diamond*}{\Diamond\varphi \Rightarrow \neg\Diamond\neg\varphi}\,\neg\mathrm{R}}{\Rightarrow \Diamond\varphi \to \neg\Diamond\neg\varphi}\,\to\mathrm{R}$$

But $\Box\varphi \lor \Box\neg\varphi$ and $\Diamond\varphi \to \neg\Diamond\neg\varphi$ are not valid in $\mathbf{K}$.

If we allowed side formulas in addition to φ in the premise, and allowed the $\Box$ rule to add $\Box$ to only φ on the right, or allowed the $\Diamond$ rule to add $\Diamond$ to only φ on the left (but do nothing to the side formulas) we would be able to deriva:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\,\neg\mathrm{R}}{\Rightarrow \neg\varphi, \varphi}\,\mathrm{XR}}{\Rightarrow \neg\varphi, \Box\varphi}\,\Box*}{\Rightarrow \neg\varphi \lor \Box\varphi}\,\lor\mathrm{R} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg\mathrm{L}}{\Diamond\neg\varphi, \varphi \Rightarrow}\,\Diamond*}{\varphi \Rightarrow \neg\Diamond\neg\varphi}\,\neg\mathrm{R}}{\Rightarrow \varphi \to \neg\Diamond\neg\varphi}\,\to\mathrm{R}$$

But $\neg\varphi \lor \Box\varphi$ (which is equivalent to $\varphi \to \Box\varphi$) and $\varphi \to \neg\Diamond\neg\varphi$ are not valid in $\mathbf{K}$.

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7.3 Sequent Derivações for K

nml:seq:prk: sec

Exemplo 7.1. We give a sequent calculus derivação that shows $\vdash (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\psi, \varphi \Rightarrow \varphi} \quad \dfrac{\psi \Rightarrow \psi}{\psi, \varphi \Rightarrow \psi}}{\psi, \varphi \Rightarrow \varphi \land \psi}\,\land\mathrm{R}}{\Box\psi, \Box\varphi \Rightarrow \Box(\varphi \land \psi)}\,\Box}{\Box\varphi \land \Box\psi, \Box\varphi \Rightarrow \Box(\varphi \land \psi)}\,\land\mathrm{L}}{\Box\varphi, \Box\varphi \land \Box\psi \Rightarrow \Box(\varphi \land \psi)}\,\mathrm{XL}}{\Box\varphi \land \Box\psi, \Box\varphi \land \Box\psi \Rightarrow \Box(\varphi \land \psi)}\,\land\mathrm{L}}{\Box\varphi \land \Box\psi \Rightarrow \Box(\varphi \land \psi)}\,\mathrm{CL}}{\Rightarrow (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)}\,\to\mathrm{R}$$

Exemplo 7.2. We give a sequent calculus derivação that shows $\vdash \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi)$.

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\doubleLine
\UnaryInfC{$\varphi \Rightarrow \varphi, \psi$}
\AxiomC{$\psi \Rightarrow \psi$}
\doubleLine
\UnaryInfC{$\psi \Rightarrow \varphi, \psi$}
\RightLabel{$\lor$L}
\BinaryInfC{$\varphi \lor \psi \Rightarrow \varphi, \psi$}
\RightLabel{$\Diamond$}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi, \Diamond\psi$}
\RightLabel{$\lor$R}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi, \Diamond\varphi \lor \Diamond\psi$}
\RightLabel{XR}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi \lor \Diamond\psi, \Diamond\varphi$}
\RightLabel{$\lor$R}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi \lor \Diamond\psi, \Diamond\varphi \lor \Diamond\psi$}
\RightLabel{CR}
\UnaryInfC{$\Diamond(\varphi \lor \psi) \Rightarrow \Diamond\varphi \lor \Diamond\psi$}
\RightLabel{$\to$R}
\UnaryInfC{$\Rightarrow \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi)$}
\end{prooftree}
$$

Here is a derivação of DUAL.

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$\neg$R}
\UnaryInfC{$\neg\varphi, \varphi \Rightarrow$}
\RightLabel{$\Diamond$}
\UnaryInfC{$\Diamond\neg\varphi, \Box\varphi \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\Box\varphi \Rightarrow \neg\Diamond\neg\varphi$}
\RightLabel{$\to$R}
\UnaryInfC{$\Rightarrow \Box\varphi \to \neg\Diamond\neg\varphi$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$\neg$R}
\UnaryInfC{$\Rightarrow \varphi, \neg\varphi$}
\RightLabel{XR}
\UnaryInfC{$\Rightarrow \neg\varphi, \varphi$}
\RightLabel{$\Box$}
\UnaryInfC{$\Rightarrow \Diamond\neg\varphi, \Box\varphi$}
\RightLabel{XR}
\UnaryInfC{$\Rightarrow \Box\varphi, \Diamond\neg\varphi$}
\RightLabel{$\neg$R}
\UnaryInfC{$\neg\Diamond\neg\varphi \Rightarrow \Box\varphi$}
\RightLabel{$\to$R}
\UnaryInfC{$\Rightarrow \neg\Diamond\neg\varphi \to \Box\varphi$}
\RightLabel{$\land$R}
\BinaryInfC{$\Rightarrow \Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$}
\end{prooftree}
$$

Problema 7.1. Find sequent calculus proofs in **K** for the following fórmulas:

1. $\Box\neg p \to \Box(p \to q)$
2. $(\Box p \lor \Box q) \to \Box(p \lor q)$
3. $\Diamond p \to \Diamond(p \lor q)$
4. $\Box(p \land q) \to \Box p$

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7.4 Rules for Other Accessibility Relations

nml:seq:mru:sec In order to deal with logics determined by special accessibility relations, we consider the additional rules in Tabela 7.1.

Adding these rules results in systems that are sound and complete for the logics given in Tabela 7.2.

Exemplo 7.3. We give a sequent derivação that shows **K4** $\vdash$ 4, O.s., $\Box\varphi \to \Box\Box\varphi$.

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\text{T}\Box \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \Diamond\varphi}\text{T}\Diamond$$

$$\frac{\Gamma \Rightarrow \Delta}{\Box\Gamma \Rightarrow \Diamond\Delta}\text{D}$$

$$\frac{\Gamma, \Diamond\Pi \Rightarrow \Box\Delta, \Lambda, \varphi}{\Box\Gamma, \Pi \Rightarrow \Delta, \Diamond\Lambda, \Box\varphi}\text{B}\Box \qquad \frac{\varphi, \Diamond\Gamma, \Pi \Rightarrow \Box\Lambda, \Delta}{\Diamond\varphi, \Gamma, \Box\Pi \Rightarrow \Lambda, \Diamond\Delta}\text{B}\Diamond$$

$$\frac{\Box\Gamma \Rightarrow \Diamond\Delta, \varphi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\varphi}4\Box \qquad \frac{\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}{\Diamond\varphi, \Box\Gamma \Rightarrow \Diamond\Delta}4\Diamond$$

$$\frac{\Box\Gamma, \Diamond\Pi \Rightarrow \Box\Delta, \Diamond\Lambda, \varphi}{\Box\Gamma, \Diamond\Pi \Rightarrow \Box\Delta, \Diamond\Lambda, \Box\varphi}5\Box \qquad \frac{\varphi, \Diamond\Gamma, \Box\Pi \Rightarrow \Diamond\Delta, \Box\Lambda}{\Diamond\varphi, \Diamond\Gamma, \Box\Pi \Rightarrow \Diamond\Delta, \Box\Lambda}5\Diamond$$

Tabela 7.1: More modal rules.

nml:seq:mru:
tab:more-rules

Logic	R is ...	Rules
T = **KT**	reflexive	$\Box$, T$\Box$, T$\Diamond$
D = **KD**	serial	$\Box$, D
K4	transitive	$\Box$, 4$\Box$, 4$\Diamond$
B = **KTB**	reflexive, symmetric	$\Box$, T$\Box$, T$\Diamond$ B$\Box$, B$\Diamond$
S4 = **KT4**	reflexive, transitive	$\Box$, T$\Box$, T$\Diamond$ 4$\Box$, 4$\Diamond$
S5 = **KT5**	reflexive, transitive, euclidean	$\Box$, T$\Box$, T$\Diamond$ 5$\Box$, 5$\Diamond$

Tabela 7.2: Sequent rules for various modal logics.

nml:seq:mru:
tab:logics-rules

$$\dfrac{\dfrac{\Box\varphi \Rightarrow \Box\varphi}{\Box\varphi \Rightarrow \Box\Box\varphi}\,4\Box}{\Rightarrow \Box\varphi \to \Box\Box\varphi}\,\to\mathrm{R}$$

Exemplo 7.4. We give a sequent derivação that shows **S5** $\vdash$ 5, O.s., $\Diamond\varphi \to \Box\Diamond\varphi$.

$$\dfrac{\dfrac{\Diamond\varphi \Rightarrow \Diamond\varphi}{\Diamond\varphi \Rightarrow \Box\Diamond\varphi}\,5\Box}{\Rightarrow \Diamond\varphi \to \Box\Diamond\varphi}\,\to\mathrm{R}$$

Exemplo 7.5. The sequent calculus for **S5** is not complete without the Cut rule; p.ex., $\Diamond\Box\varphi \to \varphi$, which is valid in **S5**, has no proof without Cut. Here is a derivação using Cut:

$$\dfrac{\dfrac{\dfrac{\Box\varphi \Rightarrow \Box\varphi}{\Diamond\Box\varphi \Rightarrow \Box\varphi}\,5\Diamond \qquad \dfrac{\varphi \Rightarrow \varphi}{\Box\varphi \Rightarrow \varphi}\,\mathrm{T}\Box}{\Diamond\Box\varphi \Rightarrow \varphi}\,\mathrm{Cut}}{\Rightarrow \Diamond\Box\varphi \to \varphi}\,\to\mathrm{R}$$

Problema 7.2. Give sequent derivações that show the following:

1. **KT5** $\vdash$ B;
2. **KT5** $\vdash$ 4;
3. **KDB4** $\vdash$ T;
4. **KB4** $\vdash$ 5;
5. **KB5** $\vdash$ 4;
6. **KT** $\vdash$ D.

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Referências Bibliográficas