

Capítulo udf

Frame Definability

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frd.1 Introduction

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One question that interests modal logicians is the relationship between the accessibility relation and the truth of certain fórmulas in models with that accessibility relation. For instance, suppose the accessibility relation is reflexive, O.s., for every $w \in W$, Rww . In other words, every world is accessible from itself. That means that when $\Box\varphi$ is true at a world w , w itself is among the accessible worlds at which φ must therefore be true. So, if the accessibility relation R of $\mathfrak{M}$ is reflexive, then whatever world w and formula φ we take, $\Box\varphi \rightarrow \varphi$ will be true there (in other words, the schema $\Box p \rightarrow p$ and all its substitution instances are true in $\mathfrak{M}$).

The converse, however, is false. It's not the case, p.ex., that if $\Box p \rightarrow p$ is true in $\mathfrak{M}$, then R is reflexive. For we can easily find a non-reflexive model $\mathfrak{M}$ where $\Box p \rightarrow p$ is true at all worlds: take the model with a single world w , not accessible from itself, but with $w \in V(p)$. By picking the truth value of p suitably, we can make $\Box\varphi \rightarrow \varphi$ true in a model that is not reflexive.

The solution is to remove the variable assignment V from the equation. If we require that $\Box p \rightarrow p$ is true at all worlds in $\mathfrak{M}$, regardless of which worlds are in $V(p)$, then it is necessary that R is reflexive. For in any non-reflexive model, there will be at least one world w such that not Rww . If we set $V(p) = W \setminus \{w\}$, then p will be true at all worlds other than w , and so at all worlds accessible from w (since w is guaranteed not to be accessible from w , and w is the only world where p is false). On the other hand, p is false at w , so $\Box p \rightarrow p$ is false at w .

This suggests that we should introduce a notation for model structures without a valuation: we call these *frames*. A frame $\mathfrak{F}$ is simply a pair $\langle W, R \rangle$ consisting of a set of worlds with an accessibility relation. Every model $\langle W, R, V \rangle$ is then, as we say, *based on* the frame $\langle W, R \rangle$. Conversely, a frame determines the class of models based on it; and a class of frames determines the class of

If R is ...	then ... is true in $\mathfrak{M}$:
<i>serial</i> : $\forall u \exists v Ruv$	$\Box p \rightarrow \Diamond p$ (D)
<i>reflexive</i> : $\forall w Rww$	$\Box p \rightarrow p$ (T)
<i>symmetric</i> : $\forall u \forall v (Ruv \rightarrow Rvu)$	$p \rightarrow \Box \Diamond p$ (B)
<i>transitive</i> : $\forall u \forall v \forall w ((Ruv \wedge Rvw) \rightarrow Ruw)$	$\Box p \rightarrow \Box \Box p$ (4)
<i>euclidean</i> : $\forall w \forall u \forall v ((Rwu \wedge Rvw) \rightarrow Ruw)$	$\Diamond p \rightarrow \Box \Diamond p$ (5)

Tabela frd.1: Five correspondence facts.

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tab:five

models which are based on any frame in the class. And we can define $\mathfrak{F} \models \varphi$, the notion of a **fórmula** being *valid* in a frame as: $\mathfrak{M} \Vdash \varphi$ for all $\mathfrak{M}$ based on $\mathfrak{F}$.

With this notation, we can establish correspondence relations between **fórmulas** and classes of frames: p.ex., $\mathfrak{F} \models \Box p \rightarrow p$ if, and only if, $\mathfrak{F}$ is reflexive.

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frd.2 Properties of Accessibility Relations

Many modal **fórmulas** turn out to be characteristic of simple, and even familiar, properties of the accessibility relation. In one direction, that means that any model that has a given property makes a corresponding **fórmula** (and all its substitution instances) true. We begin with five classical examples of kinds of accessibility relations and the formulas the truth of which they guarantee.

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Teorema frd.1. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. If R has the property on the left side of **Tabela frd.1**, every instance of the **fórmula** on the right side is true in $\mathfrak{M}$.*

nml:frd:acc:
thm:soundschemas

Prova. Here is the case for B: to show that the schema is true in a model we need to show that all of its instances are true at all worlds in the model. So let $\varphi \rightarrow \Box \Diamond \varphi$ be a given instance of B, and let $w \in W$ be an arbitrary world. Suppose the antecedent φ is true at w , in order to show that $\Box \Diamond \varphi$ is true at w . So we need to show that $\Diamond \varphi$ is true at all w' accessible from w . Now, for any w' such that Rww' we have, using the hypothesis of symmetry, that also $Rw'w$ (see **Figura frd.1**). Since $\mathfrak{M}, w \Vdash \varphi$, we have $\mathfrak{M}, w' \Vdash \Diamond \varphi$. Since w' was an arbitrary world such that Rww' , we have $\mathfrak{M}, w \Vdash \Box \Diamond \varphi$.

We leave the other cases as exercises. □

Problema frd.1. Complete the proof of **Teorema frd.1**.

Notice that the converse implications of **Teorema frd.1** do not hold: it's not true that if a model verifies a schema, then the accessibility relation of that model has the corresponding property. In the case of T and reflexive models,

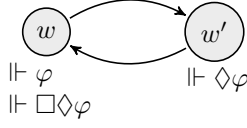


Figura frd.1: The argument from symmetry.

nml:frd:acc:
fig:Bsymm

it is easy to give an example of a model in which T itself fails: let $W = \{w\}$ and $V(p) = \emptyset$. Then R is not reflexive, but $\mathfrak{M}, w \Vdash \Box p$ and $\mathfrak{M}, w \nVdash p$. But here we have just a single instance of T that fails in $\mathfrak{M}$; other instances, p.ex., $\Box \neg p \rightarrow \neg p$ are true. It is harder to give examples where *every substitution instance* of T is true in $\mathfrak{M}$ and $\mathfrak{M}$ is not reflexive. But there are such models, too:

Proposição frd.2. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model such that $W = \{u, v\}$, where worlds u and v are related by R : O.s., both Ruv and Rvu . Suppose that for all p : $u \in V(p) \Leftrightarrow v \in V(p)$. Then:*

1. *For all φ : $\mathfrak{M}, u \Vdash \varphi$ if and only if $\mathfrak{M}, v \Vdash \varphi$ (use induction on φ).*
2. *Every instance of T is true in $\mathfrak{M}$.*

Since $\mathfrak{M}$ is not reflexive (it is, in fact, irreflexive), the converse of **Teorema frd.1** fails in the case of T (similar arguments can be given for some—though not all—the other schemas mentioned in **Teorema frd.1**).

Problema frd.2. Prove the claims in **Proposição frd.2**.

Although we will focus on the five classical fórmulas D, T, B, 4, and 5, we record in **Tabela frd.2** a few more properties of accessibility relations. The accessibility relation R is partially functional, if from every world at most one world is accessible. If it is the case that from every world exactly one world is accessible, we call it functional. (Thus the functional relations are precisely those that are both serial and partially functional). They are called “functional” because the accessibility relation operates like a (partial) function. A relation is weakly dense if whenever Ruv , there is a w “between” u and v . So weakly dense relations are in a sense the opposite of transitive relations: in a transitive relation, whenever you can reach v from u by a detour via w , you can reach v from u directly; in a weakly dense relation, whenever you can reach v from u directly, you can also reach it by a detour via some w . A relation is weakly directed if whenever you can reach worlds u and v from some world w , you can reach a single world t from both u and v —this is sometimes called the “diamond property” or “confluence.”

Problema frd.3. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. Show that if R satisfies the left-hand properties of **Tabela frd.2**, every instance of the corresponding right-hand formula is true in $\mathfrak{M}$.

If R is ...	then ... is true in $\mathfrak{M}$:
<i>partially functional</i> : $\forall w \forall u \forall v ((Rwu \wedge Ruv) \rightarrow u = v)$	$\Diamond p \rightarrow \Box p$
<i>functional</i> : $\forall w \exists v \forall u (Rwu \leftrightarrow u = v)$	$\Diamond p \leftrightarrow \Box p$
<i>weakly dense</i> : $\forall u \forall v (Ruv \rightarrow \exists w (Ruw \wedge Ruv))$	$\Box \Box p \rightarrow \Box p$
<i>weakly connected</i> : $\forall w \forall u \forall v ((Rwu \wedge Ruv) \rightarrow (Ruv \vee u = v \vee Rvu))$	$\Box((p \wedge \Box p) \rightarrow q) \vee \Box((q \wedge \Box q) \rightarrow p)$ (L)
<i>weakly directed</i> : $\forall w \forall u \forall v ((Rwu \wedge Ruv) \rightarrow \exists t (Rut \wedge Rvt))$	$\Diamond \Box p \rightarrow \Box \Diamond p$ (G)

Tabela frd.2: Five more correspondence facts.

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tab:anotherfive

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frd.3 Frames

Definição frd.3. A *frame* is a pair $\mathfrak{F} = \langle W, R \rangle$ where W is a non-empty set of worlds and R a binary relation on W . A model $\mathfrak{M}$ is *based on* a frame $\mathfrak{F} = \langle W, R \rangle$ if and only if $\mathfrak{M} = \langle W, R, V \rangle$ for some valuation V .

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sec

Definição frd.4. If $\mathfrak{F}$ is a frame, we say that φ is *valid in* $\mathfrak{F}$, $\mathfrak{F} \models \varphi$, if $\mathfrak{M} \models \varphi$ for every model $\mathfrak{M}$ based on $\mathfrak{F}$.

If $\mathcal{F}$ is a class of frames, we say φ is *valid in* $\mathcal{F}$, $\mathcal{F} \models \varphi$, iff $\mathfrak{F} \models \varphi$ for every frame $\mathfrak{F} \in \mathcal{F}$.

The reason frames are interesting is that correspondence between schemas and properties of the accessibility relation R is at the level of frames, *not of models*. For instance, although T is true in all reflexive models, not every model in which T is true is reflexive. However, it *is* true that not only is T *valid* on all reflexive *frames*, also every frame in which T is valid is reflexive.

Observação 1. Validity in a class of frames is a special case of the notion of validity in a class of models: $\mathcal{F} \models \varphi$ iff $\mathcal{C} \models \varphi$ where $\mathcal{C}$ is the class of all models based on a frame in $\mathcal{F}$.

Obviously, if a *fórmula* or a schema is valid, O.s., valid with respect to the class of *all* models, it is also valid with respect to any class $\mathcal{F}$ of frames.

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frd.4 Frame Definability

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Even though the converse implications of **Teorema frd.1** fail, they hold if we replace “model” by “frame”: for the properties considered in **Teorema frd.1**, it is true that if a **fórmula** is valid in a *frame* then the accessibility relation of that frame has the corresponding property. So, the **fórmulas** considered *define* the classes of frames that have the corresponding property.

Definição frd.5. If $\mathcal{F}$ is a class of frames, we say φ *defines* $\mathcal{F}$ iff $\mathfrak{F} \models \varphi$ for all and only frames $\mathfrak{F} \in \mathcal{F}$.

We now proceed to establish the full definability results for frames.

nml:frd:def:
thm:fullCorrespondence

Teorema frd.6. *If the **fórmula** on the right side of **Tabela frd.1** is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ has the property on the left side.*

- Prova.*
1. Suppose D is valid in $\mathfrak{F} = \langle W, R \rangle$, O.s., $\mathfrak{F} \models \Box p \rightarrow \Diamond p$. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model based on $\mathfrak{F}$, and $w \in W$. We have to show that there is a v such that Rwv . Suppose not: then both $\mathfrak{M} \models \Box \varphi$ and $\mathfrak{M}, w \not\models \Diamond \varphi$ for any φ , including p . But then $\mathfrak{M}, w \not\models \Box p \rightarrow \Diamond p$, contradicting the assumption that $\mathfrak{F} \models \Box p \rightarrow \Diamond p$.
 2. Suppose T is valid in $\mathfrak{F}$, O.s., $\mathfrak{F} \models \Box p \rightarrow p$. Let $w \in W$ be an arbitrary world; we need to show Rww . Let $u \in V(p)$ if and only if Rwu (when q is other than p , $V(q)$ is arbitrary, say $V(q) = \emptyset$). Let $\mathfrak{M} = \langle W, R, V \rangle$. By construction, for all u such that Rwu : $\mathfrak{M}, u \models p$, and hence $\mathfrak{M}, w \models \Box p$. But by hypothesis $\Box p \rightarrow p$ is true at w , so that $\mathfrak{M}, w \models p$, but by definition of V this is possible only if Rww .
 3. We prove the contrapositive: Suppose $\mathfrak{F}$ is not symmetric, we show that B, O.s., $p \rightarrow \Box \Diamond p$ is not valid in $\mathfrak{F} = \langle W, R \rangle$. If $\mathfrak{F}$ is not symmetric, there are $u, v \in W$ such that Ruv but not Rvu . Define V such that $w \in V(p)$ if and only if not Rvw (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V \rangle$. Now, by definition of V , $\mathfrak{M}, w \models p$ for all w such that not Rvw , in particular, $\mathfrak{M}, u \models p$ since not Rvu . Also, since Rvw iff $w \notin V(p)$, there is no w such that Rvw and $\mathfrak{M}, w \models p$, and hence $\mathfrak{M}, v \not\models \Diamond p$. Since Ruv , also $\mathfrak{M}, u \not\models \Box \Diamond p$. It follows that $\mathfrak{M}, u \not\models p \rightarrow \Box \Diamond p$, and so B is not valid in $\mathfrak{F}$.
 4. Suppose 4 is valid in $\mathfrak{F} = \langle W, R \rangle$, O.s., $\mathfrak{F} \models \Box p \rightarrow \Box \Box p$, and let $u, v, w \in W$ be arbitrary worlds such that Ruv and Rvw ; we need to show that Ruw . Define V such that $z \in V(p)$ if and only if Ruz (and V is arbitrary otherwise). Let $\mathfrak{M} = \langle W, R, V \rangle$. By definition of V , $\mathfrak{M}, z \models p$ for all z such that Ruz , and hence $\mathfrak{M}, u \models \Box p$. But by hypothesis 4, $\Box p \rightarrow \Box \Box p$, is true at u , so that $\mathfrak{M}, u \models \Box \Box p$. Since Ruv and Rvw , we have $\mathfrak{M}, w \models p$, but by definition of V this is possible only if Ruw , as desired.

5. We proceed contrapositively, assuming that the frame $\mathfrak{F} = \langle W, R \rangle$ is not euclidean, and show that it falsifies 5, O.s., $\mathfrak{F} \not\models \Diamond p \rightarrow \Box \Diamond p$. Suppose there are worlds $u, v, w \in W$ such that Rwu and Rwv but not Ruv . Define V such that for all worlds z , $z \in V(p)$ if and only if it is *not* the case that Ruz . Let $\mathfrak{M} = \langle W, R, V \rangle$. Then by hypothesis $\mathfrak{M}, v \Vdash p$ and since Rwv also $\mathfrak{M}, w \Vdash \Diamond p$. However, there is no world y such that Ruy and $\mathfrak{M}, y \Vdash p$ so $\mathfrak{M}, u \not\models \Diamond p$. Since Rwu , it follows that $\mathfrak{M}, w \not\models \Box \Diamond p$, so that 5, $\Diamond p \rightarrow \Box \Diamond p$, fails at w . $\square$

You'll notice a difference between the proof for D and the other cases: no mention was made of the valuation V . In effect, we proved that if $\mathfrak{M} \Vdash D$ then $\mathfrak{M}$ is serial. So D defines the class of serial *models*, not just frames.

Corolário frd.7. *Any model where D is true is serial.*

*nml:frd:def:
prop:D-serial*

Corolário frd.8. *Each fórmula on the right side of Tabela frd.1 defines the class of frames which have the property on the left side.*

Prova. In Teorema frd.1, we proved that if a model has the property on the left, the fórmula on the right is true in it. Thus, if a frame $\mathfrak{F}$ has the property on the left, the fórmula on the right is valid in $\mathfrak{F}$. In Teorema frd.6, we proved the converse implications: if a fórmula on the right is valid in $\mathfrak{F}$, $\mathfrak{F}$ has the property on the left. $\square$

Problema frd.4. Show that if the fórmula on the right side of Tabela frd.2 is valid in a frame $\mathfrak{F}$, then $\mathfrak{F}$ has the property on the left side. To do this, consider a frame that does *not* satisfy the property on the left, and define a suitable V such that the fórmula on the right is false at some world.

Teorema frd.6 also shows that the properties can be combined: for instance if both B and 4 are valid in $\mathfrak{F}$ then the frame is both symmetric and transitive, etc. Many important modal logics are characterized as the set of fórmulas valid in all frames that combine some frame properties, and so we can characterize them as the set of fórmulas valid in all frames in which the corresponding defining fórmulas are valid. For instance, the classical system **S4** is the set of all fórmulas valid in all reflexive and transitive frames, O.s., in all those where both T and 4 are valid. **S5** is the set of all formulas valid in all reflexive, symmetric, and euclidean frames, O.s., all those where all of T, B, and 5 are valid.

Logical relationships between properties of R in general correspond to relationships between the corresponding defining fórmulas. For instance, every reflexive relation is serial; hence, whenever T is valid in a frame, so is D. (Note that this relationship is *not* that of entailment. It is not the case that whenever $\mathfrak{M}, w \Vdash T$ then $\mathfrak{M}, w \Vdash D$.) We record some such relationships.

Proposição frd.9. *Let R be a binary relation on a set W ; then:*

*nml:frd:def:
prop:relation-facts*

1. *If R is reflexive, then it is serial.*

2. If R is symmetric, then it is transitive if and only if it is euclidean.
3. If R is symmetric or euclidean then it is weakly directed (it has the “diamond property”).
4. If R is euclidean then it is weakly connected.
5. If R is functional then it is serial.

Problema frd.5. Prove **Proposição frd.9**.

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frd.5 First-order Definability

nml:frd:fol:
sec

We’ve seen that a number of properties of accessibility relations of frames can be defined by modal **fórmulas**. For instance, symmetry of frames can be defined by the **fórmula** $B, p \rightarrow \Box \Diamond p$. The conditions we’ve encountered so far can all be expressed by first-order **fórmulas** in a **linguagem** involving a single two-place **símbolo de predicado**. For instance, symmetry is defined by $\forall x \forall y (Q(x, y) \rightarrow Q(y, x))$ in the sense that a first-order **estrutura** $\mathfrak{M}$ with $|\mathfrak{M}| = W$ and $Q^{\mathfrak{M}} = R$ satisfies the preceding formula iff R is symmetric. This suggests the following definition:

Definição frd.10. A class $\mathcal{F}$ of frames is *first-order definable* if there is a **sentença** φ in the first-order language with a single two-place **símbolo de predicado** Q such that $\mathfrak{F} = \langle W, R \rangle \in \mathcal{F}$ iff $\mathfrak{M} \models \varphi$ in the first-order **estrutura** $\mathfrak{M}$ with $|\mathfrak{M}| = W$ and $Q^{\mathfrak{M}} = R$.

It turns out that the properties and modal **fórmulas** that define them considered so far are exceptional. Not every **fórmula** defines a first-order definable class of frames, and not every first-order definable class of frames is definable by a modal formula.

A counterexample to the first is given by the Löb formula:

$$\Box(\Box p \rightarrow p) \rightarrow \Box p. \quad (W)$$

W defines the class of transitive and converse well-founded frames. A relation is well-founded if there is no infinite sequence $w_1, w_2, \dots$ such that $Rw_2w_1, Rw_3w_2, \dots$. For instance, the relation $<$ on $\mathbb{N}$ is well-founded, whereas the relation $<$ on $\mathbb{Z}$ is not. A relation is converse well-founded iff its converse is well-founded. So converse well-founded relations are those where there is no infinite sequence $w_1, w_2, \dots$ such that $Rw_1w_2, Rw_2w_3, \dots$.

There is, however, no first-order **fórmula** defining transitive converse well-founded relations. For suppose $\mathfrak{M} \models \beta$ iff $R = Q^{\mathfrak{M}}$ is transitive converse well-founded. Let φ_n be the **fórmula**

$$(Q(a_1, a_2) \wedge \dots \wedge Q(a_{n-1}, a_n))$$

Now consider the set of fórmulas

$$\Gamma = \{\beta, \varphi_1, \varphi_2, \dots\}.$$

Every finite subset of Γ is satisfiable: Let k be largest such that φ_k is in the subset, $|\mathfrak{M}_k| = \{1, \dots, k\}$, $a_i^{\mathfrak{M}_k} = i$, and $Q^{\mathfrak{M}_k} = <$. Since $<$ on $\{1, \dots, k\}$ is transitive and converse well-founded, $\mathfrak{M}_k \models \beta$. $\mathfrak{M}_k \models \varphi_i$ by construction, for all $i \leq k$. By the Compactness Theorem for first-order logic, Γ is satisfiable in some estrutura $\mathfrak{M}$. By hypothesis, since $\mathfrak{M} \models \beta$, the relation $Q^{\mathfrak{M}}$ is converse well-founded. But clearly, $a_1^{\mathfrak{M}}, a_2^{\mathfrak{M}}, \dots$ would form an infinite sequence of the kind ruled out by converse well-foundedness.

A counterexample to the second claim is given by the property of universality: for every u and v , Ruv . Universal frames are first-order definable by the formula $\forall x \forall y Q(x, y)$. However, no modal formula is valid in all and only the universal frames. This is a consequence of a result that is independently interesting: the formulas valid in universal frames are exactly the same as those valid in reflexive, symmetric, and transitive frames. There are reflexive, symmetric, and transitive frames that are not universal, hence every fórmula valid in all universal frames is also valid in some non-universal frames.

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frd.6 Equivalence Relations and S5

The modal logic **S5** is characterized as the set of fórmulas valid on all universal frames, O.s., every world is accessible from every world, including itself. In such a scenario, $\Box$ corresponds to necessity and $\Diamond$ to possibility: $\Box\varphi$ is true if φ is true at *every* world, and $\Diamond\varphi$ is true if φ is true at *some* world. It turns out that **S5** can also be characterized as the fórmulas valid on all reflexive, symmetric, and transitive frames, O.s., on all *equivalence relations*.

nml:frd:es5:
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Definição frd.11. A binary relation R on W is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation R on W is *universal* if and only if Ruv for all $u, v \in W$.

Since T, B, and 4 characterize the reflexive, symmetric, and transitive frames, the frames where the accessibility relation is an equivalence relation are exactly those in which all three fórmulas are valid. It turns out that the equivalence relations can also be characterized by other combinations of fórmulas, since the conditions with which we've defined equivalence relations are equivalent to combinations of other familiar conditions on R .

Proposição frd.12. *The following are equivalent:*

nml:frd:es5:
prop:equivalences

1. R is an equivalence relation;
2. R is reflexive and euclidean;
3. R is serial, symmetric, and euclidean;

4. R is serial, symmetric, and transitive.

Prova. Exercise. □

Problema frd.6. Prove **Proposição frd.12** by showing:

1. If R is symmetric and transitive, it is euclidean.
2. If R is reflexive, it is serial.
3. If R is reflexive and euclidean, it is symmetric.
4. If R is symmetric and euclidean, it is transitive.
5. If R is serial, symmetric, and transitive, it is reflexive.

Explain why this suffices for the proof that the conditions are equivalent.

Proposição frd.12 is the semantic counterpart to **??**, in that it gives an equivalent characterization of the modal logic of frames over which R is an equivalence relation (the logic traditionally referred to as **S5**).

What is the relationship between universal and equivalence relations? Although every universal relation is an equivalence relation, clearly not every equivalence relation is universal. However, the **fórmulas** valid on all universal relations are exactly the same as those valid on all equivalence relations.

Proposição frd.13. Let R be an equivalence relation, and for each $w \in W$ define the equivalence class of w as the set $[w] = \{w' \in W : Rww'\}$. Then:

1. $w \in [w]$;
2. R is universal on each equivalence class $[w]$;
3. The collection of equivalence classes partitions W into mutually exclusive and jointly exhaustive subsets.

nml:frd:es5:
prop:S5=univ

Proposição frd.14. A **fórmula** φ is valid in all frames $\mathfrak{F} = \langle W, R \rangle$ where R is an equivalence relation, if and only if it is valid in all frames $\mathfrak{F} = \langle W, R \rangle$ where R is universal. Hence, the logic of universal frames is just **S5**.

Prova. It's immediate to verify that a universal relation R on W is an equivalence. Hence, if φ is valid in all frames where R is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose ψ is a **fórmula** that fails at a world w in a model $\mathfrak{M} = \langle W, R, V \rangle$ based on a frame $\langle W, R \rangle$, where R is an equivalence on W . So $\mathfrak{M}, w \not\models \psi$. Define a model $\mathfrak{M}' = \langle W', R', V' \rangle$ as follows:

1. $W' = [w]$;
2. R' is universal on W' ;

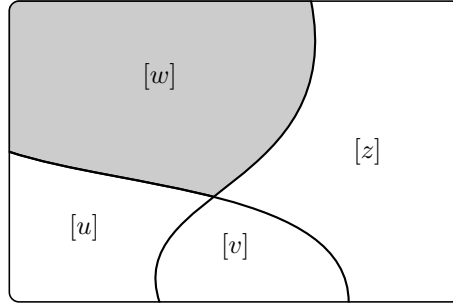


Figura frd.2: A partition of W in equivalence classes.

$$3. V'(p) = V(p) \cap W'.$$

(So the set W' of worlds in $\mathfrak{M}'$ is represented by the shaded area in **Figura frd.2**.) It is easy to see that R and R' agree on W' . Then one can show by induction on **fórmulas** that for all $w' \in W'$: $\mathfrak{M}', w' \models \varphi$ if and only if $\mathfrak{M}, w' \models \varphi$ for each φ (this makes sense since $W' \subseteq W$). In particular, $\mathfrak{M}', w \not\models \psi$, and ψ fails in a model based on a universal frame. $\square$

nml:frd:es5:
fig:partition

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frd.7 Second-order Definability

Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (O.s., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal **fórmula** is true at a world are related to first-order **fórmulas**. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place **símbolo de predicado** Q for the accessibility relation, but also a one-place **símbolo de predicado** P_i for the **variáveis proposicionais** p_i occurring in φ .

nml:frd:st:
sec

Definição frd.15. The *standard translation* $ST_x(\varphi)$ is inductively defined as follows:

1. $\varphi \equiv \perp$: $ST_x(\varphi) = \perp$.
2. $\varphi \equiv \top$: $ST_x(\varphi) = \top$.
3. $\varphi \equiv p_i$: $ST_x(\varphi) = P_i(x)$.
4. $\varphi \equiv \neg\psi$: $ST_x(\varphi) = \neg ST_x(\psi)$.
5. $\varphi \equiv (\psi \wedge \chi)$: $ST_x(\varphi) = (ST_x(\psi) \wedge ST_x(\chi))$.

6. $\varphi \equiv (\psi \vee \chi)$: $\text{ST}_x(\varphi) = (\text{ST}_x(\psi) \vee \text{ST}_x(\chi))$.
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\text{ST}_x(\varphi) = (\text{ST}_x(\psi) \rightarrow \text{ST}_x(\chi))$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\text{ST}_x(\varphi) = (\text{ST}_x(\psi) \leftrightarrow \text{ST}_x(\chi))$.
9. $\varphi \equiv \Box\psi$: $\text{ST}_x(\varphi) = \forall y (Q(x, y) \rightarrow \text{ST}_y(\psi))$.
10. $\varphi \equiv \Diamond\psi$: $\text{ST}_x(\varphi) = \exists y (Q(x, y) \wedge \text{ST}_y(\psi))$.

For instance, $\text{ST}_x(\Box p \rightarrow p)$ is $\forall y (Q(x, y) \rightarrow P(y)) \rightarrow P(x)$. Any *estrutura* for the language of $\text{ST}_x(\varphi)$ requires a domain, a two-place relation assigned to Q , and subsets of the domain assigned to the one-place *símbolos de predicado* P_i . In other words, the components of such a *estrutura* are exactly those of a model for φ : the domain is the set of worlds, the two-place relation assigned to Q is the accessibility relation, and the subsets assigned to P_i are just the assignments $V(p_i)$. It won't surprise that satisfaction of φ in a modal model and of $\text{ST}_x(\varphi)$ in the corresponding *estrutura* agree:

nml;frd:st: prop:st **Proposição frd.16.** *Let $\mathfrak{M} = \langle W, R, V \rangle$, $\mathfrak{M}'$ be the first-order *estrutura* with $|\mathfrak{M}'| = W$, $Q^{\mathfrak{M}'} = R$, and $P_i^{\mathfrak{M}'} = V(p_i)$, and $s(x) = w$. Then*

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}', s \models \text{ST}_x(\varphi)$$

Prova. By induction on φ . □

Proposição frd.17. *Suppose φ is a modal *fórmula* and $\mathfrak{F} = \langle W, R \rangle$ is a frame. Let $\mathfrak{F}'$ be the first-order *estrutura* with $|\mathfrak{F}'| = W$ and $Q^{\mathfrak{F}'} = R$, and let φ' be the second-order formula*

$$\forall X_1 \dots \forall X_n \forall x \text{ST}_x(\varphi)[X_1/P_1, \dots, X_n/P_n],$$

*where $P_1, \dots, P_n$ are all one-place *símbolos de predicado* in $\text{ST}_x(\varphi)$. Then*

$$\mathfrak{F} \models \varphi \text{ iff } \mathfrak{F}' \models \varphi'$$

Prova. $\mathfrak{F}' \models \varphi'$ iff for every *estrutura* $\mathfrak{M}'$ where $P_i^{\mathfrak{M}'} \subseteq W$ for $i = 1, \dots, n$, and for every s with $s(x) \in W$, $\mathfrak{M}', s \models \text{ST}_x(\varphi)$. By **Proposição frd.16**, that is the case iff for all models $\mathfrak{M}$ based on $\mathfrak{F}$ and every world $w \in W$, $\mathfrak{M}, w \Vdash \varphi$. O.s., $\mathfrak{F} \models \varphi$. □

Definição frd.18. A class $\mathcal{F}$ of frames is *second-order definable* if there is a *sentença* φ in the second-order language with a single two-place *símbolo de predicado* P and quantifiers only over monadic set variables such that $\mathfrak{F} = \langle W, R \rangle \in \mathcal{F}$ iff $\mathfrak{M} \models \varphi$ in the *estrutura* $\mathfrak{M}$ with $|\mathfrak{M}| = W$ and $P^{\mathfrak{M}} = R$.

Corolário frd.19. *If a class of frames is definable by a *fórmula* φ , the corresponding class of accessibility relations is definable by a monadic second-order sentence.*

Prova. The monadic second-order sentence φ' of the preceding proof has the required property. $\square$

As an example, consider again the **fórmula** $\Box p \rightarrow p$. It defines reflexivity. Reflexivity is of course first-order definable by the **sentença** $\forall x Q(x, x)$. But it is also definable by the monadic second-order **sentença**

$$\forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)).$$

This means, of course, that the two sentences are equivalent. Here's how you might convince yourself of this directly: First suppose the second-order sentence is true in a **estrutura** $\mathfrak{M}$. Since x and X are universally quantified, the remainder must hold for any $x \in W$ and set $X \subseteq W$, p.ex., the set $\{z : Rxz\}$ where $R = Q^{\mathfrak{M}}$. So, for any s with $s(x) \in W$ and $s(X) = \{z : Rxz\}$ we have $\mathfrak{M} \models \forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)$. But by the way we've picked $s(X)$ that means $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow Q(x, y)) \rightarrow Q(x, x)$, which is equivalent to $Q(x, x)$ since the antecedent is valid. Since $s(x)$ is arbitrary, we have $\mathfrak{M} \models \forall x Q(x, x)$.

Now suppose that $\mathfrak{M} \models \forall x Q(x, x)$ and show that $\mathfrak{M} \models \forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x))$. Pick any assignment s , and assume $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y))$. Let s' be the y -variant of s with $s'(y) = s(x)$; we have $\mathfrak{M}, s' \models Q(x, y) \rightarrow X(y)$, O.s., $\mathfrak{M}, s \models Q(x, x) \rightarrow X(x)$. Since $\mathfrak{M} \models \forall x Q(x, x)$, the antecedent is true, and we have $\mathfrak{M}, s \models X(x)$, which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic second-order **sentença** of the form φ' is equivalent to a first-order **sentença**. There is no effective method to decide which ones are.

Créditos das Imagens

Referências Bibliográficas