-NoValue-

mar.1 Models of PA

Any non-standard model of **TA** is also one of **PA**. We know that non-standard models of **TA** and hence of **PA** exist. We also know that such non-standard models contain non-standard "numbers," O.s., membros of the domain that are "beyond" all the standard "numbers." But how are they arranged? How many are there? We've seen that models of the weaker theory **Q** can contain as few as a single non-standard number. But these simple estruturas are not models of **PA** or **TA**. explanation

The key to understanding the structure of models of **PA** or **TA** is to see what facts are derivável in these theories. For instance, already **PA** proves that $\forall x\, x \neq x'$ and $\forall x\, \forall y\, (x + y) = (y + x)$, so this rules out simple structures (in which these sentenças are false) as models of **PA**.

Suppose $\mathfrak{M}$ is a model of **PA**. Then if $\mathbf{PA} \vdash \varphi$, $\mathfrak{M} \vDash \varphi$. Let's again use $\mathbf{z}$ for $\mathsf{0}^{\mathfrak{M}}$, $*$ for ${\prime}^{\mathfrak{M}}$, $\oplus$ for $+^{\mathfrak{M}}$, $\otimes$ for $\times^{\mathfrak{M}}$, and $\lessdot$ for $<^{\mathfrak{M}}$. Any sentença φ then states some condition about $\mathbf{z}$, $*$, $\oplus$, $\otimes$, and $\lessdot$, and if $\mathfrak{M} \vDash \varphi$ that condition must be satisfied. For instance, if $\mathfrak{M} \vDash Q_1$, O.s., $\mathfrak{M} \vDash \forall x\, \forall y\, (x' = y' \rightarrow x = y)$, then $*$ must be injetiva.

Proposição mar.1. *In $\mathfrak{M}$, $\lessdot$ is a linear strict order, O.s., it satisfies:*

1. *Not $x \lessdot x$ for any $x \in |\mathfrak{M}|$.*
2. *If $x \lessdot y$ and $y \lessdot z$ then $x \lessdot z$.*
3. *For any $x \neq y$, $x \lessdot y$ or $y \lessdot x$*

Prova. **PA** proves:

1. $\forall x\, \neg x < x$
2. $\forall x\, \forall y\, \forall z\, ((x < y \land y < z) \rightarrow x < z)$
3. $\forall x\, \forall y\, ((x < y \lor y < x) \lor x = y))$ □

mod:mar:mpa: prop:M-discrete **Proposição mar.2.** *$\mathbf{z}$ is the least membro of $|\mathfrak{M}|$ in the $\lessdot$-ordering. For any x, $x \lessdot x^*$, and x^* is the $\lessdot$-least membro with that property. For any x, there is a unique y such that $y^* = x$. (We call y the "predecessor" of x in $\mathfrak{M}$, and denote it by ${}^*x$.)*

Prova. Exercise. □

Problema mar.1. Find sentenças in $\mathcal{L}_A$ derivável in **PA** (and hence true in $\mathfrak{N}$) which guarantee the properties of $\mathbf{z}$, $*$, and $\lessdot$ in Proposição mar.2

Proposição mar.3. *All standard membros of $\mathfrak{M}$ are less than (according to $\lessdot$) all non-standard membros.*

Prova. We'll use n as short for $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$, a standard membro of $\mathfrak{M}$. Already $\mathbf{Q}$ proves that, for any $n \in \mathbb{N}$, $\forall x\,(x < \overline{n}' \rightarrow (x = \overline{0} \lor x = \overline{1} \lor \dots \lor x = \overline{n}))$. There are no membros that are $\olessthan \mathbf{z}$. So if n is standard and x is non-standard, we cannot have $x \olessthan n$. By definition, a non-standard element is one that isn't $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for any $n \in \mathbb{N}$, so $x \neq n$ as well. Since $\olessthan$ is a linear order, we must have $n \olessthan x$. □

Proposição mar.4. *Every nonstandard membro x of $|\mathfrak{M}|$ is an element of the subset*

$$\dots^{***}x \olessthan {}^{**}x \olessthan {}^{*}x \olessthan x \olessthan x^{*} \olessthan x^{**} \olessthan x^{***} \olessthan \dots$$

We call this subset the block *of x and write it as $[x]$. It has no least and no greatest membro. It can be characterized as the set of those $y \in |\mathfrak{M}|$ such that, for some standard n, $x \oplus n = y$ or $y \oplus n = x$.*

Prova. Clearly, such a set $[x]$ always exists since every membro y of $|\mathfrak{M}|$ has a unique successor y^* and unique predecessor ${}^*y$. For successive membros y, y^* we have $y \olessthan y^*$ and y^* is the $\olessthan$-least membro of $|\mathfrak{M}|$ such that y is $\olessthan$-less than it. Since always ${}^*y \olessthan y$ and $y \olessthan y^*$, $[x]$ has no least or greatest membro. If $y \in [x]$ then $x \in [y]$, for then either $y^{*\dots*} = x$ or $x^{*\dots*} = y$. If $y^{*\dots*} = x$ (with n $*$'s), then $y \oplus n = x$ and conversely, since $\mathbf{PA} \vdash \forall x\, x'^{\dots\prime} = (x + \overline{n})$ (if n is the number of $\prime$'s). □

Proposição mar.5. *If $[x] \neq [y]$ and $x \olessthan y$, then for any $u \in [x]$ and any $v \in [y]$, $u \olessthan v$.*

Prova. Note that $\mathbf{PA} \vdash \forall x\,\forall y\,(x < y \rightarrow (x' < y \lor x' = y))$. Thus, if $u \olessthan v$, we also have $u \oplus n^* \olessthan v$ for any n if $[u] \neq [v]$.

Any $u \in [x]$ is $\olessthan y$: $x \olessthan y$ by assumption. If $u \olessthan x$, $u \olessthan y$ by transitivity. And if $x \olessthan u$ but $u \in [x]$, we have $u = x \oplus n^*$ for some n, and so $u \olessthan y$ by the fact just proved.

Now suppose that $v \in [y]$ is $\olessthan y$, O.s., $v \oplus m^* = y$ for some standard m. This rules out $v \olessthan x$, otherwise $y = v \oplus m^* \olessthan x$. Clearly also, $x \neq v$, otherwise $x \oplus m^* = v \oplus m^* = y$ and we would have $[x] = [y]$. So, $x \olessthan v$. But then also $x \oplus n^* \olessthan v$ for any n. Hence, if $x \olessthan u$ and $u \in [x]$, we have $u \olessthan v$. If $u \olessthan x$ then $u \olessthan v$ by transitivity.

Lastly, if $y \olessthan v$, $u \olessthan v$ since, as we've shown, $u \olessthan y$ and $y \olessthan v$. □

Corolário mar.6. *If $[x] \neq [y]$, $[x] \cap [y] = \emptyset$.*

Prova. Suppose $z \in [x]$ and $x \olessthan y$. Then $z \olessthan u$ for all $u \in [y]$. If $z \in [y]$, we would have $z \olessthan z$. Similarly if $y \olessthan x$. □

explanation

This means that the blocks themselves can be ordered in a way that respects $\olessthan$: $[x] \olessthan [y]$ iff $x \olessthan y$, or, equivalently, if $u \olessthan v$ for any $u \in [x]$ and $v \in [y]$. Clearly, the standard block $[0]$ is the least block. It intersects with no non-standard block, and no two non-standard blocks intersect either. Specifically, you cannot "reach" a different block by taking repeated successors or predecessors.

Proposição mar.7. *If x and y are non-standard, then $x \lessdot x \oplus y$ and $x \oplus y \notin [x]$.*

Prova. If y is nonstandard, then $y \neq \mathbf{z}$. $\mathbf{PA} \vdash \forall x\, (y \neq \mathsf{o} \rightarrow x < (x + y))$. Now suppose $x \oplus y \in [x]$. Since $x \lessdot x \oplus y$, we would have $x \oplus n^* = x \oplus y$. But $\mathbf{PA} \vdash \forall x\, \forall y\, \forall z\, ((x + y) = (x + z) \rightarrow y = z)$ (the cancellation law for addition). This would mean $y = n^*$ for some standard n; but y is assumed to be non-standard. □

Proposição mar.8. *There is no least non-standard block.*

Prova. $\mathbf{PA} \vdash \forall x\, \exists y\, ((y + y) = x \lor (y + y)' = x)$, O.s., that every x is divisible by 2 (possibly with remainder 1). If x is non-standard, so is y. By the preceding proposition, $y \lessdot y \oplus y$ and $y \oplus y \notin [y]$. Then also $y \lessdot (y \oplus y)^*$ and $(y \oplus y)^* \notin [y]$. But $x = y \oplus y$ or $x = (y \oplus y)^*$, so $y \lessdot x$ and $y \notin [x]$. □

Proposição mar.9. *There is no largest block.*

Prova. Exercise. □

Problema mar.2. Show that in a non-standard model of **PA**, there is no largest block.

mod:mar:mpa: prop:blocks-dense **Proposição mar.10.** *The ordering of the blocks is dense. That is, if $x \lessdot y$ and $[x] \neq [y]$, then there is a block $[z]$ distinct from both that is between them.*

Prova. Suppose $x \lessdot y$. As before, $x \oplus y$ is divisible by two (possibly with remainder): there is a $z \in |\mathfrak{M}|$ such that either $x \oplus y = z \oplus z$ or $x \oplus y = (z \oplus z)^*$. The element z is the "average" of x and y, and $x \lessdot z$ and $z \lessdot y$. □

Problema mar.3. Write out a detailed proof of Proposição mar.10. Which sentença must **PA** deriva in order to guarantee the existence of z? Why is $x \lessdot z$ and $z \lessdot y$, and why is $[x] \neq [z]$ and $[z] \neq [y]$?

The non-standard blocks are therefore ordered like the rationals: they form a infinito enumerável dense linear ordering without endpoints. One can show that any two such infinito enumerável orderings are isomorphic. It follows that for any two enumerável non-standard models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of true arithmetic, their reducts to the language containing $<$ and $=$ only are isomorphic. Indeed, an isomorphism h can be defined as follows: the standard parts of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic to the standard model $\mathfrak{N}$ and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore isomorphic; an isomorphism of the blocks can be extended to an isomorphism *within* the blocks by matching up arbitrary elements in each, and then taking the image of the successor of x in $\mathfrak{M}_1$ to be the successor of the image of x in $\mathfrak{M}_2$. Note that it does *not* follow that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic in the full language of arithmetic (indeed, isomorphism is always relative to a linguagem), as there are non-isomorphic ways to define addition explanation and multiplication over $|\mathfrak{M}_1|$ and $|\mathfrak{M}_2|$. (This also follows from a famous theorem due to Vaught that the number of countable models of a complete theory cannot be 2.)

Créditos das Imagens

Referências Bibliográficas

