Capítulo udf

Models of Arithmetic

-NoValue-

mar.1 Introduction

The *standard model* of arithmetic is the estrutura $\mathfrak{N}$ with $|\mathfrak{N}| = \mathbb{N}$ in which o, $\prime$, $+$, $\times$, and $<$ are interpreted as you would expect. That is, o is 0, $\prime$ is the successor function, $+$ is interpreted as addition and $\times$ as multiplication of the numbers in $\mathbb{N}$. Specifically,

$$\begin{aligned}
\mathsf{o}^{\mathfrak{N}} &= 0 \\
\prime^{\mathfrak{N}}(n) &= n + 1 \\
+^{\mathfrak{N}}(n, m) &= n + m \\
\times^{\mathfrak{N}}(n, m) &= nm
\end{aligned}$$

Of course, there are structures for $\mathcal{L}_A$ that have domains other than $\mathbb{N}$. For instance, we can take $\mathfrak{M}$ with domain $|\mathfrak{M}| = \{a\}^*$ (the finite sequences of the single symbol a, O.s., $\emptyset$, a, aa, aaa, ...), and interpretations

$$\begin{aligned}
\mathsf{o}^{\mathfrak{M}} &= \emptyset \\
\prime^{\mathfrak{M}}(s) &= s \frown a \\
+^{\mathfrak{M}}(n, m) &= a^{n+m} \\
\times^{\mathfrak{M}}(n, m) &= a^{nm}
\end{aligned}$$

These two structures are "essentially the same" in the sense that the only difference is the membros of the domínios but not how the membros of the domínios are related among each other by the interpretation functions. We say that the two estruturas are *isomorphic*.

It is an easy consequence of the compactness theorem that any theory true in $\mathfrak{N}$ also has models that are not isomorphic to $\mathfrak{N}$. Such structures are called *non-standard*. The interesting thing about them is that while the membros of a standard model (O.s., $\mathfrak{N}$, but also all estruturas isomorphic to it) are exhausted by the values of the standard numerals $\overline{n}$, O.s.,

$$|\mathfrak{N}| = \{\mathrm{Val}^{\mathfrak{N}}(\overline{n}) : n \in \mathbb{N}\}$$

that isn't the case in non-standard models: if $\mathfrak{M}$ is non-standard, then there is at least one $x \in |\mathfrak{M}|$ such that $x \neq \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all n.

These non-standard elements are pretty neat: they are "infinite natural numbers." But their existence also explains, in a sense, the incompleteness phenomena. Consider an example, p.ex., the consistency statement for Peano arithmetic, $\mathsf{Con}_{\mathbf{PA}}$, O.s., $\lnot\exists x\, \mathsf{Prf}_{\mathbf{PA}}(x, \ulcorner\bot\urcorner)$. Since **PA** neither proves $\mathsf{Con}_{\mathbf{PA}}$ nor $\lnot\mathsf{Con}_{\mathbf{PA}}$, either can be consistently added to **PA**. Since **PA** is consistent, $\mathfrak{N} \vDash \mathsf{Con}_{\mathbf{PA}}$, and consequently $\mathfrak{N} \nvDash \lnot\mathsf{Con}_{\mathbf{PA}}$. So $\mathfrak{N}$ is *not* a model of $\mathbf{PA} \cup \{\lnot\mathsf{Con}_{\mathbf{PA}}\}$, and all its models must be nonstandard. Models of $\mathbf{PA} \cup \{\lnot\mathsf{Con}_{\mathbf{PA}}\}$ must contain some membro that serves as the witness that makes $\exists x\, \mathsf{Prf}_{\mathbf{PA}}(\ulcorner\bot\urcorner)$ true, O.s., a Gödel number of a derivação of a contradiction from **PA**. Such um membro can't be standard—since $\mathbf{PA} \vdash \lnot\mathsf{Prf}_{\mathbf{PA}}(\overline{n}, \ulcorner\bot\urcorner)$ for every n.

-NoValue-

mar.2 Standard Models of Arithmetic

The language of arithmetic $\mathcal{L}_A$ is obviously intended to be about numbers, specifically, about natural numbers. So, "the" standard model $\mathfrak{N}$ is special: it is the model we want to talk about. But in logic, we are often just interested in structural properties, and any two estruturas that are isomorphic share those. So we can be a bit more liberal, and consider any estrutura that is isomorphic to $\mathfrak{N}$ "standard." mod:mar:stm:sec

Definição mar.1. A estrutura for $\mathcal{L}_A$ is *standard* if it is isomorphic to $\mathfrak{N}$.

Proposição mar.2. *If a estrutura $\mathfrak{M}$ is standard, then its domain is the set of values of the standard numerals, O.s.,* mod:mar:stm:prop:standard-domain

$$|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$$

Prova. Clearly, every $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \in |\mathfrak{M}|$. We just have to show that every $x \in |\mathfrak{M}|$ is equal to $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some n. Since $\mathfrak{M}$ is standard, it is isomorphic to $\mathfrak{N}$. Suppose $g \colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism. Then $g(n) = g(\mathrm{Val}^{\mathfrak{N}}(\overline{n})) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. But for every $x \in |\mathfrak{M}|$, there is an $n \in \mathbb{N}$ such that $g(n) = x$, since g is sobrejetiva. $\square$

explanation If a structure $\mathfrak{M}$ for $\mathcal{L}_A$ is standard, the elements of its domínio can all be named by the standard numerals $\overline{0}$, $\overline{1}$, $\overline{2}$, ..., O.s., the terms o, o', o'', etc. Of course, this does not mean that the membros of $|\mathfrak{M}|$ *are* the numbers, just that we can pick them out the same way we can pick out the numbers in $|\mathfrak{N}|$.

Problema mar.1. Show that the converse of Proposição mar.2 is false, O.s., give an example of a estrutura $\mathfrak{M}$ with $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$ that is not isomorphic to $\mathfrak{N}$.

mod:mar:stm: prop:thq-standard **Proposição mar.3.** *If $\mathfrak{M} \vDash \mathbf{Q}$, and $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$, then $\mathfrak{M}$ is standard.*

Prova. We have to show that $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$. Consider the function $g \colon \mathbb{N} \to |\mathfrak{M}|$ defined by $g(n) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. By the hypothesis, g is sobrejetiva. It is also injetiva: $\mathbf{Q} \vdash \overline{n} \neq \overline{m}$ whenever $n \neq m$. Thus, since $\mathfrak{M} \vDash \mathbf{Q}$, $\mathfrak{M} \vDash \overline{n} \neq \overline{m}$, whenever $n \neq m$. Thus, if $n \neq m$, then $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \neq \mathrm{Val}^{\mathfrak{M}}(\overline{m})$, O.s., $g(n) \neq g(m)$.

We also have to verify that g is an isomorphism.

1. We have $g(\mathsf{o}^{\mathfrak{N}}) = g(0)$ since, $\mathsf{o}^{\mathfrak{N}} = 0$. By definition of g, $g(0) = \mathrm{Val}^{\mathfrak{M}}(\overline{0})$. But $\overline{0}$ is just o, and the value of a term which happens to be a símbolo de constante is given by what the estrutura assigns to that símbolo de constante, O.s., $\mathrm{Val}^{\mathfrak{M}}(\mathsf{o}) = \mathsf{o}^{\mathfrak{M}}$. So we have $g(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$ as required.

2. $g({\prime}^{\mathfrak{N}}(n)) = g(n+1)$, since $\prime$ in $\mathfrak{N}$ is the successor function on $\mathbb{N}$. Then, $g(n+1) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+1})$ by definition of g. But $\overline{n+1}$ is the same term as $\overline{n}'$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n}')$. By the definition of the value function, this is $= {\prime}^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ we get $g({\prime}^{\mathfrak{N}}(n)) = {\prime}^{\mathfrak{M}}(g(n))$.

3. $g(+^{\mathfrak{N}}(n,m)) = g(n+m)$, since $+$ in $\mathfrak{N}$ is the addition function on $\mathbb{N}$. Then, $g(n+m) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+m})$ by definition of g. But $\mathbf{Q} \vdash \overline{n+m} = (\overline{n} + \overline{m})$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n+m}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n} + \overline{m})$. By the definition of the value function, this is $= +^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ and $\mathrm{Val}^{\mathfrak{M}}(\overline{m}) = g(m)$, we get $g(+^{\mathfrak{N}}(n,m)) = +^{\mathfrak{M}}(g(n), g(m))$.

4. $g(\times^{\mathfrak{N}}(n,m)) = \times^{\mathfrak{M}}(g(n), g(m))$: Exercise.

5. $\langle n, m\rangle \in <^{\mathfrak{N}}$ iff $n < m$. If $n < m$, then $\mathbf{Q} \vdash \overline{n} < \overline{m}$, and also $\mathfrak{M} \vDash \overline{n} < \overline{m}$. Thus $\langle \mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m})\rangle \in <^{\mathfrak{M}}$, O.s., $\langle g(n), g(m)\rangle \in <^{\mathfrak{M}}$. If $n \not< m$, then $\mathbf{Q} \vdash \neg \overline{n} < \overline{m}$, and consequently $\mathfrak{M} \nvDash \overline{n} < \overline{m}$. Thus, as before, $\langle g(n), g(m)\rangle \notin <^{\mathfrak{M}}$. Together, we get: $\langle n, m\rangle \in <^{\mathfrak{N}}$ iff $\langle g(n), g(m)\rangle \in <^{\mathfrak{M}}$. □

The function g is the most obvious way of defining a mapping from $\mathbb{N}$ to the domain of any other estrutura $\mathfrak{M}$ for $\mathcal{L}_A$, since every such $\mathfrak{M}$ contains membros named by $\overline{0}$, $\overline{1}$, $\overline{2}$, etc. So it isn't surprising that if $\mathfrak{M}$ makes at least some basic statements about the $\overline{n}$'s true in the same way that $\mathfrak{N}$ does, and g is also bijective, then g will turn into an isomorphism. In fact, if $|\mathfrak{M}|$ contains no membros other than what the $\overline{n}$'s name, it's the only one. explanation

mod:mar:stm: prop:thq-unique-iso **Proposição mar.4.** *If $\mathfrak{M}$ is standard, then g from the proof of Proposição mar.3 is the only isomorphism from $\mathfrak{N}$ to $\mathfrak{M}$.*

Prova. Suppose $h\colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism between $\mathfrak{N}$ and $\mathfrak{M}$. We show that $g = h$ by induction on n. If $n = 0$, then $g(0) = \mathsf{o}^{\mathfrak{M}}$ by definition of g. But since h is an isomorphism, $h(0) = h(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$, so $g(0) = h(0)$.

Now consider the case for $n + 1$. We have

$$
\begin{aligned}
g(n+1) &= \mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) \text{ by definition of } g\\
&= \mathrm{Val}^{\mathfrak{M}}(\overline{n}') \text{ since } \overline{n+1} \equiv \overline{n}'\\
&= {\prime}^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n})) \text{ by definition of } \mathrm{Val}^{\mathfrak{M}}(t')\\
&= {\prime}^{\mathfrak{M}}(g(n)) \text{ by definition of } g\\
&= {\prime}^{\mathfrak{M}}(h(n)) \text{ by induction hypothesis}\\
&= h({\prime}^{\mathfrak{N}}(n)) \text{ since } h \text{ is an isomorphism}\\
&= h(n+1)
\end{aligned}
$$

□

explanation For any infinito enumerável set M, there's uma bijeção between $\mathbb{N}$ and M, so every such set M is potentially the domínio of a standard model $\mathfrak{M}$. In fact, once you pick an object $z \in M$ and a suitable function s as $\mathsf{o}^{\mathfrak{M}}$ and ${\prime}^{\mathfrak{M}}$, the interpretations of $+$, $\times$, and $<$ is already fixed. Only functions $s\colon M \to M \setminus \{z\}$ that are both injetiva and sobrejetiva are suitable in a standard model as ${\prime}^{\mathfrak{M}}$. The range of s cannot contain z, since otherwise $\forall x\, \mathsf{o} \neq x'$ would be false. That sentença is true in $\mathfrak{N}$, and so $\mathfrak{M}$ also has to make it true. The function s has to be injetiva, since the successor function ${\prime}^{\mathfrak{N}}$ in $\mathfrak{N}$ is, and that ${\prime}^{\mathfrak{N}}$ is injetiva is expressed by a sentença true in $\mathfrak{N}$. It has to be sobrejetiva because otherwise there would be some $x \in M \setminus \{z\}$ not in the domain of s, O.s., the sentença $\forall x\,(x = \mathsf{o} \lor \exists y\, y' = x)$ would be false in $\mathfrak{M}$—but it is true in $\mathfrak{N}$.

-NoValue-

mar.3 Non-Standard Models

explanation We call a estrutura for $\mathcal{L}_A$ standard if it is isomorphic to $\mathfrak{N}$. If a estrutura isn't isomorphic to $\mathfrak{N}$, it is called non-standard.

Definição mar.5. A estrutura $\mathfrak{M}$ for $\mathcal{L}_A$ is *non-standard* if it is not isomorphic to $\mathfrak{N}$. The membros $x \in |\mathfrak{M}|$ which are equal to $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some $n \in \mathbb{N}$ are called *standard numbers* (of $\mathfrak{M}$), and those not, *non-standard numbers.*

explanation By Proposição mar.2, any standard estrutura for $\mathcal{L}_A$ contains only standard membros. Consequently, a non-standard estrutura must contain at least one non-standard element. In fact, the existence of a non-standard membro guarantees that the estrutura is non-standard.

Proposição mar.6. *If a estrutura $\mathfrak{M}$ for $\mathcal{L}_A$ contains a non-standard number, $\mathfrak{M}$ is non-standard.*

Prova. Suppose not, O.s., suppose $\mathfrak{M}$ standard but contains a non-standard number x. Let $g\colon \mathbb{N} \to |\mathfrak{M}|$ be an isomorphism. It is easy to see (by induction on n) that $g(\mathrm{Val}^{\mathfrak{N}}(\overline{n})) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. In other words, g maps standard numbers of $\mathfrak{N}$ to standard numbers of $\mathfrak{M}$. If $\mathfrak{M}$ contains a non-standard number, g cannot be sobrejetiva, contrary to hypothesis. □

Problema mar.2. Recall that **Q** contains the axioms

$$\forall x\, \forall y\, (x' = y' \to x = y) \tag{Q_1}$$

$$\forall x\, \mathsf{o} \neq x' \tag{Q_2}$$

$$\forall x\, (x = \mathsf{o} \lor \exists y\, x = y') \tag{Q_3}$$

Give estruturas $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ such that

1. $\mathfrak{M}_1 \vDash Q_1$, $\mathfrak{M}_1 \vDash Q_2$, $\mathfrak{M}_1 \nvDash Q_3$;
2. $\mathfrak{M}_2 \vDash Q_1$, $\mathfrak{M}_2 \nvDash Q_2$, $\mathfrak{M}_2 \vDash Q_3$; and
3. $\mathfrak{M}_3 \nvDash Q_1$, $\mathfrak{M}_3 \vDash Q_2$, $\mathfrak{M}_3 \vDash Q_3$;

Obviously, you just have to specify $\mathsf{o}^{\mathfrak{M}_i}$ and $\prime^{\mathfrak{M}_i}$ for each.

It is easy enough to specify non-standard estruturas for $\mathcal{L}_A$. For instance, take the structure with domínio $\mathbb{Z}$ and interpret all non-logical symbols as usual. Since negative numbers are not values of $\overline{n}$ for any n, this structure is non-standard. Of course, it will not be a *model* of arithmetic in the sense that it makes the same sentences true as $\mathfrak{N}$. For instance, $\forall x\, x' \neq \mathsf{o}$ is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem. explanation

Proposição mar.7. *Let* $\mathbf{TA} = \{\varphi : \mathfrak{N} \vDash \varphi\}$ *be the theory of* $\mathfrak{N}$. **TA** *has a enumerável non-standard model.*

Prova. Expand $\mathcal{L}_A$ by a new símbolo de constante c and consider the set of sentenças

$$\Gamma = \mathbf{TA} \cup \{c \neq \overline{0}, c \neq \overline{1}, c \neq \overline{2}, \dots\}$$

Any model $\mathfrak{M}^c$ of Γ would contain um membro $x = c^{\mathfrak{M}}$ which is non-standard, since $x \neq \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all $n \in \mathbb{N}$. Also, obviously, $\mathfrak{M}^c \vDash \mathbf{TA}$, since $\mathbf{TA} \subseteq \Gamma$. If we turn $\mathfrak{M}^c$ into a estrutura $\mathfrak{M}$ for $\mathcal{L}_A$ simply by forgetting about c, its domain still contains the non-standard x, and also $\mathfrak{M} \vDash \mathbf{TA}$. The latter is guaranteed since c does not occur in **TA**. So, it suffices to show that Γ has a model.

We use the compactness theorem to show that Γ has a model. If every finite subset of Γ is satisfiable, so is Γ. Consider any finite subset $\Gamma_0 \subseteq \Gamma$. Γ_0 includes some sentenças of **TA** and some of the form $c \neq \overline{n}$, but only finitely many. Suppose k is the largest number so that $c \neq \overline{k} \in \Gamma_0$. Define $\mathfrak{N}_k$ by expanding $\mathfrak{N}$ to include the interpretation $c^{\mathfrak{N}_k} = k + 1$. $\mathfrak{N}_k \vDash \Gamma_0$: if $\varphi \in \mathbf{TA}$, $\mathfrak{N}_k \vDash \varphi$ since $\mathfrak{N}_k$ is just like $\mathfrak{N}$ in all respects except c, and c does not occur in φ. And $\mathfrak{N}_k \vDash c \neq \overline{n}$, since $n \leq k$, and $\mathrm{Val}^{\mathfrak{N}_k}(c) = k + 1$. Thus, every finite subset of Γ is satisfiable. □

-NoValue-

mar.4 Models of Q

explanation We know that there are non-standard estruturas that make the same sentenças true as $\mathfrak{N}$ does, O.s., is a model of **TA**. Since $\mathfrak{N} \vDash \mathbf{Q}$, any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, p.ex., $\mathbf{Q} \nvDash \forall x \, \forall y \, (x + y) = (y + x)$. Weaker theories are easier to satisfy: they have more models. P.ex., **Q** has models which make $\forall x \, \forall y \, (x + y) = (y + x)$ false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly.

Exemplo mar.8. Consider the estrutura $\mathfrak{K}$ with domain $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations mod:mar:mdq: ex:model-K-of-Q

$$\mathsf{o}^{\mathfrak{K}} = 0$$

$$\prime^{\mathfrak{K}}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases}$$

$$+^{\mathfrak{K}}(x, y) = \begin{cases} x+y & \text{if } x,\, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases}$$

$$\times^{\mathfrak{K}}(x, y) = \begin{cases} xy & \text{if } x,\, y \in \mathbb{N} \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases}$$

$$<^{\mathfrak{K}} = \{\langle x, y\rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a\rangle : x \in |\mathfrak{K}|\}$$

To show that $\mathfrak{K} \vDash \mathbf{Q}$ we have to verify that all axioms of **Q** are true in $\mathfrak{K}$. For convenience, let's write x^* for $\prime^{\mathfrak{K}}(x)$ (the "successor" of x in $\mathfrak{K}$), $x \oplus y$ for $+^{\mathfrak{K}}(x, y)$ (the "sum" of x and y in $\mathfrak{K}$, $x \otimes y$ for $\times^{\mathfrak{K}}(x, y)$ (the "product" of x and y in $\mathfrak{K}$), and $x \lessdot y$ for $\langle x, y\rangle \in <^{\mathfrak{K}}$. With these abbreviations, we can give the operations in $\mathfrak{K}$ more perspicuously as

x	x^*
n	$n+1$
a	a

$x \oplus y$	0	m	a
0	0	m	a
n	n	$n+m$	a
a	a	a	a

$x \otimes y$	0	m	a
0	0	0	0
n	0	nm	a
a	0	a	a

We have $n \lessdot m$ iff $n < m$ for n, $m \in \mathbb{N}$ and $x \lessdot a$ for all $x \in |\mathfrak{K}|$.

$\mathfrak{K} \vDash \forall x \, \forall y \, (x' = y' \rightarrow x = y)$ since $*$ is injetiva. $\mathfrak{K} \vDash \forall x \, \mathsf{o} \neq x'$ since 0 is not a $*$-successor in $\mathfrak{K}$. $\mathfrak{K} \vDash \forall x \, (x = \mathsf{o} \lor \exists y \, x = y')$ since for every $n > 0$, $n = (n-1)^*$, and $a = a^*$.

$\mathfrak{K} \vDash \forall x \, (x + \mathsf{o}) = x$ since $n \oplus 0 = n + 0 = n$, and $a \oplus 0 = a$ by definition of $\oplus$. $\mathfrak{K} \vDash \forall x \, \forall y \, (x + y') = (x + y)'$ is a bit trickier. If n, m are both standard, we have:

$$(n \oplus m^*) = (n + (m + 1)) = (n + m) + 1 = (n \oplus m)^*$$

since $\oplus$ and * agree with $+$ and $'$ on standard numbers. Now suppose $x \in |\mathfrak{K}|$. Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if $y \in |\mathfrak{K}|$ but $x = a$. Here we also have to distinguish cases according to whether $y = n$ is standard or $y = b$:

$$\begin{aligned}(a \oplus n^*) &= (a \oplus (n+1)) = a = a^* = (a \oplus n)^* \\ (a \oplus a^*) &= (a \oplus a) = a = a^* = (a \oplus a)^*\end{aligned}$$

This is of course a bit more detailed than needed. For instance, since $a \oplus z = a$ whatever z is, we can immediately conclude $a \oplus a^* = a$. The remaining axioms can be verified the same way.

$\mathfrak{K}$ is thus a model of $\mathbf{Q}$. Its "addition" $\oplus$ is also commutative. But there are other sentences true in $\mathfrak{N}$ but false in $\mathfrak{K}$, and vice versa. For instance, $a \mathbin{\lessdot} a$, so $\mathfrak{K} \vDash \exists x\, x < x$ and $\mathfrak{K} \nvDash \forall x\, \neg x < x$. This shows that $\mathbf{Q} \nvdash \forall x\, \neg x < x$.

Problema mar.3. Prove that $\mathfrak{K}$ from Exemplo mar.8 satisfies the remaining axioms of $\mathbf{Q}$,

$$\forall x\, (x \times \mathsf{o}) = \mathsf{o} \tag{Q_6}$$

$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \tag{Q_7}$$

$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (z' + x) = y) \tag{Q_8}$$

Find a sentença only involving $'$ true in $\mathfrak{N}$ but false in $\mathfrak{K}$.

mod:mar:mdq: ex:model-L-of-Q **Exemplo mar.9.** Consider the estrutura $\mathfrak{L}$ with domain $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$ and interpretations $'^{\mathfrak{L}} = *$, $+^{\mathfrak{L}} = \oplus$ given by

x	x^*
n	$n+1$
a	a
b	b

$x \oplus y$	m	a	b
n	$n+m$	b	a
a	a	b	a
b	b	b	a

Since $*$ is injetiva, 0 is not in its range, and every $x \in |\mathfrak{L}|$ other than 0 is, axioms Q_1–Q_3 are true in $\mathfrak{L}$. For any x, $x \oplus 0 = x$, so Q_4 is true as well. For Q_5, consider $x \oplus y^*$ and $(x \oplus y)^*$. They are equal if x and y are both standard, since then $*$ and $\oplus$ agree with $'$ and $+$. If x is non-standard, and y is standard, we have $x \oplus y^* = x = x^* = (x \oplus y)^*$. If x and y are both non-standard, we have four cases:

$$\begin{aligned}a \oplus a^* &= b = b^* = (a \oplus a)^* \\ b \oplus b^* &= a = a^* = (b \oplus b)^* \\ b \oplus a^* &= b = b^* = (b \oplus y)^* \\ a \oplus b^* &= a = a^* = (a \oplus b)^*\end{aligned}$$

If x is standard, but y is non-standard, we have

$$n \oplus a^* = n \oplus a = b = b^* = (n \oplus a)^*$$
$$n \oplus b^* = n \oplus b = a = a^* = (n \oplus b)^*$$

So, $\mathfrak{L} \vDash Q_5$. However, $a \oplus 0 \neq 0 \oplus a$, so $\mathfrak{L} \nvDash \forall x\, \forall y\, (x + y) = (y + x)$.

Problema mar.4. Expand $\mathfrak{L}$ of Exemplo mar.9 to include $\otimes$ and $\lessdot$ that interpret $\times$ and $<$. Show that your structure satisfies the remaining axioms of **Q**,

$$\forall x\, (x \times \mathsf{o}) = \mathsf{o} \tag{Q_6}$$
$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \tag{Q_7}$$
$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (z' + x) = y) \tag{Q_8}$$

Problema mar.5. In $\mathfrak{L}$ of Exemplo mar.9, $a^* = a$ and $b^* = b$. Is there a model of **Q** in which $a^* = b$ and $b^* = a$?

explanation We've explicitly constructed models of **Q** in which the non-standard membros live "beyond" the standard elements. In fact, that much is required by the axioms. A non-standard membro x cannot be $\lessdot\, 0$, since $\mathbf{Q} \vdash \forall x\, \neg x < 0$ (see **??**). Also, for every n, $\mathbf{Q} \vdash \forall x\, (x < \overline{n}' \rightarrow (x = \overline{0} \lor x = \overline{1} \lor \cdots \lor x = \overline{n}))$ (**??**), so we can't have $a \lessdot n$ for any $n > 0$.

-NoValue-

mar.5 Models of PA

explanation Any non-standard model of **TA** is also one of **PA**. We know that non-standard models of **TA** and hence of **PA** exist. We also know that such non-standard models contain non-standard "numbers," O.s., membros of the domain that are "beyond" all the standard "numbers." But how are they arranged? How many are there? We've seen that models of the weaker theory **Q** can contain as few as a single non-standard number. But these simple estruturas are not models of **PA** or **TA**.

The key to understanding the structure of models of **PA** or **TA** is to see what facts are derivável in these theories. For instance, already **PA** proves that $\forall x\, x \neq x'$ and $\forall x\, \forall y\, (x + y) = (y + x)$, so this rules out simple structures (in which these sentenças are false) as models of **PA**.

Suppose $\mathfrak{M}$ is a model of **PA**. Then if $\mathbf{PA} \vdash \varphi$, $\mathfrak{M} \vDash \varphi$. Let's again use $\mathbf{z}$ for $\mathsf{o}^{\mathfrak{M}}$, $*$ for $\prime^{\mathfrak{M}}$, $\oplus$ for $+^{\mathfrak{M}}$, $\otimes$ for $\times^{\mathfrak{M}}$, and $\lessdot$ for $<^{\mathfrak{M}}$. Any sentença φ then states some condition about $\mathbf{z}$, $*$, $\oplus$, $\otimes$, and $\lessdot$, and if $\mathfrak{M} \vDash \varphi$ that condition must be satisfied. For instance, if $\mathfrak{M} \vDash Q_1$, O.s., $\mathfrak{M} \vDash \forall x\, \forall y\, (x' = y' \rightarrow x = y)$, then $*$ must be injetiva.

Proposição mar.10. *In $\mathfrak{M}$, $\lessdot$ is a linear strict order, O.s., it satisfies:*

1. *Not $x \lessdot x$ for any $x \in |\mathfrak{M}|$.*

2. *If $x \lessdot y$ and $y \lessdot z$ then $x \lessdot z$.*

3. *For any $x \neq y$, $x \lessdot y$ or $y \lessdot x$*

Prova. **PA** proves:

1. $\forall x \, \neg x < x$

2. $\forall x \, \forall y \, \forall z \, ((x < y \land y < z) \rightarrow x < z)$

3. $\forall x \, \forall y \, ((x < y \lor y < x) \lor x = y))$ □

mod:mar:mpa: prop:M-discrete **Proposição mar.11.** *$\mathbf{z}$ is the least membro of $|\mathfrak{M}|$ in the $\lessdot$-ordering. For any x, $x \lessdot x^*$, and x^* is the $\lessdot$-least membro with that property. For any x, there is a unique y such that $y^* = x$. (We call y the "predecessor" of x in $\mathfrak{M}$, and denote it by ${}^*x$.)*

Prova. Exercise. □

Problema mar.6. Find sentenças in $\mathcal{L}_A$ derivável in **PA** (and hence true in $\mathfrak{N}$) which guarantee the properties of $\mathbf{z}$, $*$, and $\lessdot$ in Proposição mar.11

Proposição mar.12. *All standard membros of $\mathfrak{M}$ are less than (according to $\lessdot$) all non-standard membros.*

Prova. We'll use n as short for $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$, a standard membro of $\mathfrak{M}$. Already **Q** proves that, for any $n \in \mathbb{N}$, $\forall x \, (x < \overline{n}' \rightarrow (x = \overline{0} \lor x = \overline{1} \lor \cdots \lor x = \overline{n}))$. There are no membros that are $\lessdot \mathbf{z}$. So if n is standard and x is non-standard, we cannot have $x \lessdot n$. By definition, a non-standard element is one that isn't $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for any $n \in \mathbb{N}$, so $x \neq n$ as well. Since $\lessdot$ is a linear order, we must have $n \lessdot x$. □

Proposição mar.13. *Every nonstandard membro x of $|\mathfrak{M}|$ is an element of the subset*

$$\ldots {}^{***}x \lessdot {}^{**}x \lessdot {}^{*}x \lessdot x \lessdot x^* \lessdot x^{**} \lessdot x^{***} \lessdot \ldots$$

We call this subset the block *of x and write it as $[x]$. It has no least and no greatest membro. It can be characterized as the set of those $y \in |\mathfrak{M}|$ such that, for some standard n, $x \oplus n = y$ or $y \oplus n = x$.*

Prova. Clearly, such a set $[x]$ always exists since every membro y of $|\mathfrak{M}|$ has a unique successor y^* and unique predecessor ${}^*y$. For successive membros y, y^* we have $y \lessdot y^*$ and y^* is the $\lessdot$-least membro of $|\mathfrak{M}|$ such that y is $\lessdot$-less than it. Since always ${}^*y \lessdot y$ and $y \lessdot y^*$, $[x]$ has no least or greatest membro. If $y \in [x]$ then $x \in [y]$, for then either $y^{*\cdots*} = x$ or $x^{*\cdots*} = y$. If $y^{*\cdots*} = x$ (with n $*$'s), then $y \oplus n = x$ and conversely, since $\mathbf{PA} \vdash \forall x \, x'^{\cdots\prime} = (x + \overline{n})$ (if n is the number of $\prime$'s). □

Proposição mar.14. *If $[x] \neq [y]$ and $x \olessthan y$, then for any $u \in [x]$ and any $v \in [y]$, $u \olessthan v$.*

Prova. Note that $\mathbf{PA} \vdash \forall x\, \forall y\, (x < y \rightarrow (x' < y \lor x' = y))$. Thus, if $u \olessthan v$, we also have $u \oplus n^* \olessthan v$ for any n if $[u] \neq [v]$.

Any $u \in [x]$ is $\olessthan y$: $x \olessthan y$ by assumption. If $u \olessthan x$, $u \olessthan y$ by transitivity. And if $x \olessthan u$ but $u \in [x]$, we have $u = x \oplus n^*$ for some n, and so $u \olessthan y$ by the fact just proved.

Now suppose that $v \in [y]$ is $\olessthan y$, O.s., $v \oplus m^* = y$ for some standard m. This rules out $v \olessthan x$, otherwise $y = v \oplus m^* \olessthan x$. Clearly also, $x \neq v$, otherwise $x \oplus m^* = v \oplus m^* = y$ and we would have $[x] = [y]$. So, $x \olessthan v$. But then also $x \oplus n^* \olessthan v$ for any n. Hence, if $x \olessthan u$ and $u \in [x]$, we have $u \olessthan v$. If $u \olessthan x$ then $u \olessthan v$ by transitivity.

Lastly, if $y \olessthan v$, $u \olessthan v$ since, as we've shown, $u \olessthan y$ and $y \olessthan v$. □

Corolário mar.15. *If $[x] \neq [y]$, $[x] \cap [y] = \emptyset$.*

Prova. Suppose $z \in [x]$ and $x \olessthan y$. Then $z \olessthan u$ for all $u \in [y]$. If $z \in [y]$, we would have $z \olessthan z$. Similarly if $y \olessthan x$. □

explanation This means that the blocks themselves can be ordered in a way that respects $\olessthan$: $[x] \olessthan [y]$ iff $x \olessthan y$, or, equivalently, if $u \olessthan v$ for any $u \in [x]$ and $v \in [y]$. Clearly, the standard block $[0]$ is the least block. It intersects with no non-standard block, and no two non-standard blocks intersect either. Specifically, you cannot "reach" a different block by taking repeated successors or predecessors.

Proposição mar.16. *If x and y are non-standard, then $x \olessthan x \oplus y$ and $x \oplus y \notin [x]$.*

Prova. If y is nonstandard, then $y \neq \mathbf{z}$. $\mathbf{PA} \vdash \forall x\, (y \neq \mathsf{o} \rightarrow x < (x + y))$. Now suppose $x \oplus y \in [x]$. Since $x \olessthan x \oplus y$, we would have $x \oplus n^* = x \oplus y$. But $\mathbf{PA} \vdash \forall x\, \forall y\, \forall z\, ((x + y) = (x + z) \rightarrow y = z)$ (the cancellation law for addition). This would mean $y = n^*$ for some standard n; but y is assumed to be non-standard. □

Proposição mar.17. *There is no least non-standard block.*

Prova. $\mathbf{PA} \vdash \forall x\, \exists y\, ((y + y) = x \lor (y + y)' = x)$, O.s., that every x is divisible by 2 (possibly with remainder 1). If x is non-standard, so is y. By the preceding proposition, $y \olessthan y \oplus y$ and $y \oplus y \notin [y]$. Then also $y \olessthan (y \oplus y)^*$ and $(y \oplus y)^* \notin [y]$. But $x = y \oplus y$ or $x = (y \oplus y)^*$, so $y \olessthan x$ and $y \notin [x]$. □

Proposição mar.18. *There is no largest block.*

Prova. Exercise. □

Problema mar.7. Show that in a non-standard model of **PA**, there is no largest block.

mod:mar:mpa: prop:blocks-dense **Proposição mar.19.** *The ordering of the blocks is dense. That is, if $x \mathbin{\lessdot} y$ and $[x] \neq [y]$, then there is a block $[z]$ distinct from both that is between them.*

Prova. Suppose $x \mathbin{\lessdot} y$. As before, $x \oplus y$ is divisible by two (possibly with remainder): there is a $z \in |\mathfrak{M}|$ such that either $x \oplus y = z \oplus z$ or $x \oplus y = (z \oplus z)^*$. The element z is the "average" of x and y, and $x \mathbin{\lessdot} z$ and $z \mathbin{\lessdot} y$. □

Problema mar.8. Write out a detailed proof of Proposição mar.19. Which sentença must **PA** deriva in order to guarantee the existence of z? Why is $x \mathbin{\lessdot} z$ and $z \mathbin{\lessdot} y$, and why is $[x] \neq [z]$ and $[z] \neq [y]$?

The non-standard blocks are therefore ordered like the rationals: they form a infinito enumerável dense linear ordering without endpoints. One can show that any two such infinito enumerável orderings are isomorphic. It follows that for any two enumerável non-standard models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of true arithmetic, their reducts to the language containing $<$ and $=$ only are isomorphic. Indeed, an isomorphism h can be defined as follows: the standard parts of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic to the standard model $\mathfrak{N}$ and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore isomorphic; an isomorphism of the blocks can be extended to an isomorphism *within* the blocks by matching up arbitrary elements in each, and then taking the image of the successor of x in $\mathfrak{M}_1$ to be the successor of the image of x in $\mathfrak{M}_2$. Note that it does *not* follow that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic in the full language of arithmetic (indeed, isomorphism is always relative to a linguagem), as there are non-isomorphic ways to define addition and multiplication over $|\mathfrak{M}_1|$ and $|\mathfrak{M}_2|$. (This also follows from a famous theorem due to Vaught that the number of countable models of a complete theory cannot be 2.) explanation

-NoValue-

mar.6 Computable Models of Arithmetic

The standard model $\mathfrak{N}$ has two nice features. Its domain is the natural numbers $\mathbb{N}$, O.s., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral $\overline{n}$ actually picks out n. The other nice feature is that the interpretations of the non-logical symbols of $\mathcal{L}_A$ are all *computable*. The successor, addition, and multiplication functions which serve as ${\prime}^{\mathfrak{N}}$, $+^{\mathfrak{N}}$, and $\times^{\mathfrak{N}}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on $\mathfrak{N}$, O.s., $<^{\mathfrak{N}}$, is decidable. explanation

Non-standard models of arithmetical theories such as **Q** and **PA** must contain non-standard elements. Thus their domains typically include membros in addition to $\mathbb{N}$. However, any countable estrutura can be built on any infinito enumerável set, including $\mathbb{N}$. So there are also non-standard models with domain $\mathbb{N}$. In such models $\mathfrak{M}$, of course, at least some numbers cannot play the roles they usually play, since some k must be different from $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all $n \in \mathbb{N}$.

Definição mar.20. A estrutura $\mathfrak{M}$ for $\mathcal{L}_A$ is *computable* iff $|\mathfrak{M}| = \mathbb{N}$ and $\prime^{\mathfrak{M}}$, $+^{\mathfrak{M}}$, $\times^{\mathfrak{M}}$ are computable functions and $<^{\mathfrak{M}}$ is a decidable relation.

Exemplo mar.21. Recall the structure $\mathfrak{K}$ from Exemplo mar.8. Its domain was $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations mod:mar:cmp: ex:comp-model-q

$$\begin{aligned}
\mathsf{o}^{\mathfrak{K}} &= 0 \\
\prime^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\
+^{\mathfrak{K}}(x,y) &= \begin{cases} x+y & \text{if } x,\, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
\times^{\mathfrak{K}}(x,y) &= \begin{cases} xy & \text{if } x,\, y \in \mathbb{N} \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases} \\
<^{\mathfrak{K}} &= \{\langle x,y\rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x,a\rangle : n \in |\mathfrak{K}|\}
\end{aligned}$$

But $|\mathfrak{K}|$ is infinito enumerável and so is equinumerous with $\mathbb{N}$. For instance, $g\colon \mathbb{N} \to |\mathfrak{K}|$ with $g(0) = a$ and $g(n) = n+1$ for $n > 0$ is uma bijeção. We can turn it into an isomorphism between a new model $\mathfrak{K}'$ of $\mathbf{Q}$ and $\mathfrak{K}$. In $\mathfrak{K}'$, we have to assign different functions and relations to the symbols of $\mathcal{L}_A$, since different membros of $\mathbb{N}$ play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of a, not of the smallest standard number. The smallest standard number is now 1. So we assign $\mathsf{o}^{\mathfrak{K}'} = 1$. The successor function is also different now: given a standard number, O.s., an $n > 0$, it still returns $n+1$. But 0 now plays the role of a, which is its own successor. So $\prime^{\mathfrak{K}'}(0) = 0$. For addition and multiplication we likewise have

$$\begin{aligned}
+^{\mathfrak{K}'}(x,y) &= \begin{cases} x+y-1 & \text{if } x,\, y > 0 \\ 0 & \text{otherwise} \end{cases} \\
\times^{\mathfrak{K}'}(x,y) &= \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ xy - x - y + 2 & \text{if } x,\, y > 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

And we have $\langle x,y\rangle \in <^{\mathfrak{K}'}$ iff $x < y$ and $x > 0$ and $y > 0$, or if $y = 0$.

All of these functions are computable functions of natural numbers and $<^{\mathfrak{K}'}$ is a decidable relation on $\mathbb{N}$—but they are not the same functions as successor, addition, and multiplication on $\mathbb{N}$, and $<^{\mathfrak{K}'}$ is not the same relation as $<$ on $\mathbb{N}$.

Problema mar.9. Give a estrutura $\mathfrak{L}'$ with $|\mathfrak{L}'| = \mathbb{N}$ isomorphic to $\mathfrak{L}$ of Exemplo mar.9.

Exemplo mar.21 shows that $\mathbf{Q}$ has computable non-standard models with domain $\mathbb{N}$. However, the following result shows that this is not true for models of $\mathbf{PA}$ (and thus also for models of $\mathbf{TA}$). explanation

Teorema mar.22 (Tennenbaum's Theorem). $\mathfrak{N}$ *is the only computable model of* $\mathbf{PA}$.

Créditos das Imagens

Referências Bibliográficas