-NoValue-

lin.1 Compactness and Löwenheim–Skolem Properties

mod:lin:lsp: sec We now give the obvious extensions of compactness and Löwenheim–Skolem to the case of abstract logics.

Definição lin.1. An abstract logic $\langle L, \models_L\rangle$ has the *Compactness Property* if each set Γ of $L(\mathcal{L})$-sentenças is satisfiable whenever each finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

Definição lin.2. $\langle L, \models_L\rangle$ has the *Downward Löwenheim–Skolem property* if any satisfiable Γ has a enumerável model.

The notion of partial isomorphism from **??** is purely "algebraic" (O.s., given without reference to the sentenças of the language but only to the constants provided by the linguagem $\mathcal{L}$ of the estruturas), and hence it applies to the case of abstract logics. In case of first-order logic, we know from **??** that if two estruturas are partially isomorphic then they are elementarily equivalent. That proof does not carry over to abstract logics, for induction on fórmulas need not be available for arbitrary $\alpha \in L(\mathcal{L})$, but the theorem is true nonetheless, provided the Löwenheim–Skolem property holds.

mod:lin:lsp: thm:abstract-p-isom **Teorema lin.3.** *Suppose $\langle L, \models_L\rangle$ is a normal logic with the Löwenheim–Skolem property. Then any two estruturas that are partially isomorphic are elementarily equivalent in $\langle L, \models_L\rangle$.*

Prova. Suppose $\mathfrak{M} \simeq_p \mathfrak{N}$, but for some α also $\mathfrak{M} \models_L \alpha$ while $\mathfrak{N} \not\models_L \alpha$. By the Isomorphism Property we can assume that $|\mathfrak{M}|$ and $|\mathfrak{N}|$ are disjoint, and by the Expansion Property we can assume that $\alpha \in L(\mathcal{L})$ for a finite linguagem $\mathcal{L}$. Let $\mathcal{I}$ be a set of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$, and with no loss of generality also assume that if $p \in \mathcal{I}$ and $q \subseteq p$ then also $q \in \mathcal{I}$.

$|\mathfrak{M}|^{<\omega}$ is the set of finite sequences of membros of $|\mathfrak{M}|$. Let S be the ternary relation over $|\mathfrak{M}|^{<\omega}$ representing concatenation, O.s., if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ then $S(\mathbf{a}, \mathbf{b}, \mathbf{c})$ holds if and only if $\mathbf{c}$ is the concatenation of $\mathbf{a}$ and $\mathbf{b}$; and let T be the ternary relation such that $T(\mathbf{a}, b, \mathbf{c})$ holds for $b \in M$ and $\mathbf{a}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ if and only if $\mathbf{a} = a_1, \dots a_n$ and $\mathbf{c} = a_1, \dots a_n, b$. Pick new 3-place símbolos de predicado P and Q and form the estrutura $\mathfrak{M}^*$ having the universe $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, having $\mathfrak{M}$ as a substructure, and interpreting P and Q by the concatenation relations S and T (so $\mathfrak{M}^*$ is in the linguagem $\mathcal{L} \cup \{P, Q\}$).

Define $|\mathfrak{N}|^{<\omega}$, S', T', P', Q' and $\mathfrak{N}^*$ analogously. Since by hypothesis $\mathfrak{M} \simeq_p \mathfrak{N}$, there is a relation I between $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ such that $I(\mathbf{a}, \mathbf{b})$ holds if and only if $\mathbf{a}$ and $\mathbf{b}$ are isomorphic and satisfy the back-and-forth condition of **??**. Now, let $\mathfrak{M}$ be the estrutura whose domínio is the union of the domínios of $\mathfrak{M}^*$ and $\mathfrak{N}^*$, having $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as subestruturas, in the linguagem with one extra binary símbolo de predicado R interpreted by the relation I and símbolos de predicado denoting the domínios $|\mathfrak{M}|^*$ and $|\mathfrak{N}|*$.

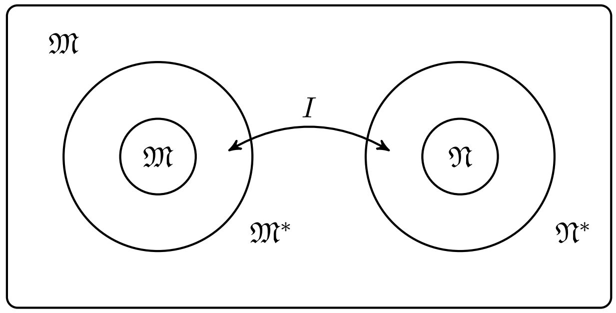

Figura 1: The estrutura $\mathfrak{M}$ with the internal partial isomorphism.

The crucial observation is that in the linguagem of the estrutura $\mathfrak{M}$ there is a *first-order* sentença θ_1 true in $\mathfrak{M}$ saying that $\mathfrak{M} \models_L \alpha$ and $\mathfrak{N} \not\models_L \alpha$ (this requires the Relativization Property), as well as a *first-order* sentença θ_2 true in $\mathfrak{M}$ saying that $\mathfrak{M} \simeq_p \mathfrak{N}$ via the partial isomorphism I. By the Löwenheim–Skolem Property, θ_1 and θ_2 are jointly true in a enumerável model $\mathfrak{M}_0$ containing partially isomorphic substructures $\mathfrak{M}_0$ and $\mathfrak{N}_0$ such that $\mathfrak{M}_0 \models_L \alpha$ and $\mathfrak{N}_0 \not\models_L \alpha$. But enumerável partially isomorphic estruturas are in fact isomorphic by **??**, contradicting the Isomorphism Property of normal abstract logics. □

Créditos das Imagens

Referências Bibliográficas