Capítulo udf

Lindström's Theorem

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lin.1 Introduction

In this chapter we aim to prove Lindström's characterization of first-order logic as the maximal logic for which (given certain further constraints) the Compactness and the Downward Löwenheim–Skolem theorems hold (**??** and **??**). First, we need a more general characterization of the general class of logics to which the theorem applies. We will restrict ourselves to *relational* languages, O.s., languages which only contain símbolos de predicado and individual constants, but no símbolos de função.

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lin.2 Abstract Logics

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Definição lin.1. An *abstract logic* is a pair $\langle L, \models_L \rangle$, where L is a function that assigns to each linguagem $\mathcal{L}$ a set $L(\mathcal{L})$ of sentenças, and $\models_L$ is a relation between estruturas for the linguagem $\mathcal{L}$ and membros of $L(\mathcal{L})$. In particular, $\langle F, \models \rangle$ is ordinary first-order logic, O.s., F is the function assigning to the linguagem $\mathcal{L}$ the set of first-order sentenças built from the constants in $\mathcal{L}$, and $\models$ is the satisfaction relation of first-order logic.

Notice that we are still employing the same notion of estrutura for a given linguagem as for first-order logic, but we do not presuppose that sentenças are build up from the basic symbols in $\mathcal{L}$ in the usual way, nor that the relation $\models_L$ is recursively defined in the same way as for first-order logic. So for instance the definition, being completely general, is intended to capture the case where sentenças in $\langle L, \models_L \rangle$ contain infinitely long conjunctions or disjunction, or quantifiers other than $\exists$ and $\forall$ (p.ex., "there are infinitely many x such that ..."), or perhaps infinitely long quantifier prefixes. To emphasize that "sentenças" in $L(\mathcal{L})$ need not be ordinary sentenças of first-order logic, in this chapter we use variáveis α, β, ... to range over them, and reserve φ, ψ, ... for ordinary first-order fórmulas.

Definição lin.2. Let $\mathrm{Mod}_L(\alpha)$ denote the class $\{\mathfrak{M} : \mathfrak{M} \models_L \alpha\}$. If the linguagem needs to be made explicit, we write $\mathrm{Mod}^{\mathcal{L}}_L(\alpha)$. Two estruturas $\mathfrak{M}$ and $\mathfrak{N}$ for $\mathcal{L}$ are *elementarily equivalent in* $\langle L, \models_L\rangle$, written $\mathfrak{M} \equiv_L \mathfrak{N}$, if the same sentenças from $L(\mathcal{L})$ are true in each.

Definição lin.3. An abstract logic $\langle L, \models_L\rangle$ for the linguagem $\mathcal{L}$ is *normal* if it satisfies the following properties:

1. (*L-Monotonicity*) For linguagens $\mathcal{L}$ and $\mathcal{L}'$, if $\mathcal{L} \subseteq \mathcal{L}'$, then $L(\mathcal{L}) \subseteq L(\mathcal{L}')$.

2. (*Expansion Property*) For each $\alpha \in L(\mathcal{L})$ there is a *finite* subset $\mathcal{L}'$ of $\mathcal{L}$ such that the relation $\mathfrak{M} \models_L \alpha$ depends only on the reduct of $\mathfrak{M}$ to $\mathcal{L}'$; O.s., if $\mathfrak{M}$ and $\mathfrak{N}$ have the same reduct to $\mathcal{L}'$ then $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{N} \models_L \alpha$.

3. (*Isomorphism Property*) If $\mathfrak{M} \models_L \alpha$ and $\mathfrak{M} \simeq \mathfrak{N}$ then also $\mathfrak{N} \models_L \alpha$.

4. (*Renaming Property*) The relation $\models_L$ is preserved under renaming: if the linguagem $\mathcal{L}'$ is obtained from $\mathcal{L}$ by replacing each symbol P by a symbol P' of the same arity and each constant c by a distinct constant c', then for each estrutura $\mathfrak{M}$ and sentença α, $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{M}' \models_L \alpha'$, where $\mathfrak{M}'$ is the $\mathcal{L}'$-estrutura corresponding to $\mathcal{L}$ and $\alpha' \in L(\mathcal{L}')$.

5. (*Boolean Property*) The abstract logic $\langle L, \models_L\rangle$ is closed under the Boolean connectives in the sense that for each $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that $\mathfrak{M} \models_L \beta$ if and only if $\mathfrak{M} \not\models_L \alpha$, and for each α and β there is a γ such that $\mathrm{Mod}_L(\gamma) = \mathrm{Mod}_L(\alpha) \cap \mathrm{Mod}_L(\beta)$. Similarly for atomic fórmulas and the other connectives.

6. (*Quantifier Property*) For each constant c in $\mathcal{L}$ and $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that
$$\mathrm{Mod}^{\mathcal{L}'}_L(\beta) = \{\mathfrak{M} : (\mathfrak{M}, a) \in \mathrm{Mod}^{\mathcal{L}}_L(\alpha) \text{ for some } a \in |\mathfrak{M}|\},$$
where $\mathcal{L}' = \mathcal{L} \setminus \{c\}$ and $(\mathfrak{M}, a)$ is the expansion of $\mathfrak{M}$ to $\mathcal{L}$ assigning a to c.

7. (*Relativization Property*) Given a sentença $\alpha \in L(\mathcal{L})$ and symbols R, c_1, ..., c_n not in $\mathcal{L}$, there is a sentença $\beta \in L(\mathcal{L} \cup \{R, c_1, \dots, c_n\})$ called the *relativization* of α to $R(x, c_1, \dots c_n)$, such that for each estrutura $\mathfrak{M}$:
$$(\mathfrak{M}, X, b_1, \dots, b_n) \models_L \beta \text{ if and only if } \mathfrak{N} \models_L \alpha,$$
where $\mathfrak{N}$ is the substructure of $\mathfrak{M}$ with domínio $|\mathfrak{N}| = \{a \in |\mathfrak{M}| : R^{\mathfrak{M}}(a, b_1, \dots, b_n)\}$ (see **??**), and $(\mathfrak{M}, X, b_1, \dots, b_n)$ is the expansion of $\mathfrak{M}$ interpreting R, c_1, ..., c_n by X, b_1, ..., b_n, respectively (with $X \subseteq M^{n+1}$).

Definição lin.4. Given two abstract logics $\langle L_1, \models_{L_1}\rangle$ and $\langle L_2, \models_{L_2}\rangle$ we say that the latter is *at least as expressive* as the former, written $\langle L_1, \models_{L_1}\rangle \leq \langle L_2, \models_{L_2}\rangle$, if for each linguagem $\mathcal{L}$ and sentença $\alpha \in L_1(\mathcal{L})$ there is a sentença $\beta \in L_2(\mathcal{L})$ such that $\mathrm{Mod}^{\mathcal{L}}_{L_1}(\alpha) = \mathrm{Mod}^{\mathcal{L}}_{L_2}(\beta)$. The logics $\langle L_1, \models_{L_1}\rangle$ and $\langle L_2, \models_{L_2}\rangle$ are *equivalent* if $\langle L_1, \models_{L_1}\rangle \leq \langle L_2, \models_{L_2}\rangle$ and $\langle L_2, \models_{L_2}\rangle \leq \langle L_1, \models_{L_1}\rangle$.

Observação 1. First-order logic, O.s., the abstract logic $\langle F, \models\rangle$, is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to extensionality, and that the relativization of a sentença α to $R(x, c_1, \dots, c_n)$ is obtained by replacing each subfórmula $\forall x\, \beta$ by $\forall x\, (R(x, c_1, \dots, c_n) \rightarrow \beta)$. Moreover, if $\langle L, \models_L\rangle$ is normal, then $\langle F, \models\rangle \leq \langle L, \models_L\rangle$, as can be can shown by induction on first-order fórmulas. Accordingly, with no loss in generality, we can assume that every first-order sentença belongs to every normal logic.

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lin.3 Compactness and Löwenheim–Skolem Properties

mod:lin:lsp: sec We now give the obvious extensions of compactness and Löwenheim–Skolem to the case of abstract logics.

Definição lin.5. An abstract logic $\langle L, \models_L\rangle$ has the *Compactness Property* if each set Γ of $L(\mathcal{L})$-sentenças is satisfiable whenever each finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

Definição lin.6. $\langle L, \models_L\rangle$ has the *Downward Löwenheim–Skolem property* if any satisfiable Γ has a enumerável model.

The notion of partial isomorphism from **??** is purely "algebraic" (O.s., given without reference to the sentenças of the language but only to the constants provided by the linguagem $\mathcal{L}$ of the estruturas), and hence it applies to the case of abstract logics. In case of first-order logic, we know from **??** that if two estruturas are partially isomorphic then they are elementarily equivalent. That proof does not carry over to abstract logics, for induction on fórmulas need not be available for arbitrary $\alpha \in L(\mathcal{L})$, but the theorem is true nonetheless, provided the Löwenheim–Skolem property holds.

mod:lin:lsp: thm:abstract-p-isom **Teorema lin.7.** *Suppose $\langle L, \models_L\rangle$ is a normal logic with the Löwenheim–Skolem property. Then any two estruturas that are partially isomorphic are elementarily equivalent in $\langle L, \models_L\rangle$.*

Prova. Suppose $\mathfrak{M} \simeq_p \mathfrak{N}$, but for some α also $\mathfrak{M} \models_L \alpha$ while $\mathfrak{N} \not\models_L \alpha$. By the Isomorphism Property we can assume that $|\mathfrak{M}|$ and $|\mathfrak{N}|$ are disjoint, and by the Expansion Property we can assume that $\alpha \in L(\mathcal{L})$ for a finite linguagem $\mathcal{L}$. Let $\mathcal{I}$ be a set of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$, and with no loss of generality also assume that if $p \in \mathcal{I}$ and $q \subseteq p$ then also $q \in \mathcal{I}$.

$|\mathfrak{M}|^{<\omega}$ is the set of finite sequences of membros of $|\mathfrak{M}|$. Let S be the ternary relation over $|\mathfrak{M}|^{<\omega}$ representing concatenation, O.s., if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ then $S(\mathbf{a}, \mathbf{b}, \mathbf{c})$ holds if and only if $\mathbf{c}$ is the concatenation of $\mathbf{a}$ and $\mathbf{b}$; and let T be the ternary relation such that $T(\mathbf{a}, b, \mathbf{c})$ holds for $b \in M$ and $\mathbf{a}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ if and only if $\mathbf{a} = a_1, \dots a_n$ and $\mathbf{c} = a_1, \dots a_n, b$. Pick new 3-place símbolos de predicado P and Q and form the estrutura $\mathfrak{M}^*$ having the universe $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, having $\mathfrak{M}$ as a substructure, and interpreting P and Q by the concatenation relations S and T (so $\mathfrak{M}^*$ is in the linguagem $\mathcal{L} \cup \{P, Q\}$).

Define $|\mathfrak{N}|^{<\omega}$, S', T', P', Q' and $\mathfrak{N}^*$ analogously. Since by hypothesis $\mathfrak{M} \simeq_p \mathfrak{N}$, there is a relation I between $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ such that $I(\mathbf{a}, \mathbf{b})$ holds if and only if $\mathbf{a}$ and $\mathbf{b}$ are isomorphic and satisfy the back-and-forth condition of **??**. Now, let $\mathfrak{M}$ be the estrutura whose domínio is the union of the domínios of $\mathfrak{M}^*$ and $\mathfrak{N}^*$, having $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as subestruturas, in the linguagem with one extra binary símbolo de predicado R interpreted by the relation I and símbolos de predicado denoting the domínios $|\mathfrak{M}|^*$ and $|\mathfrak{N}|*$.

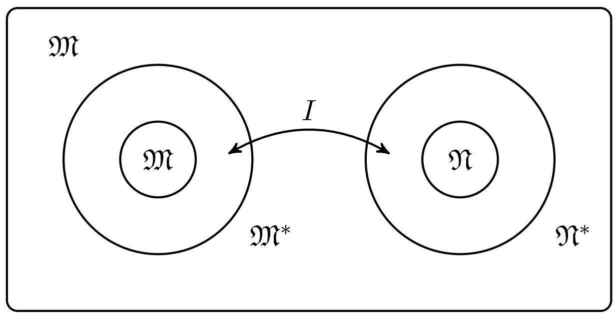

Figura lin.1: The estrutura $\mathfrak{M}$ with the internal partial isomorphism.

The crucial observation is that in the linguagem of the estrutura $\mathfrak{M}$ there is a *first-order* sentença θ_1 true in $\mathfrak{M}$ saying that $\mathfrak{M} \models_L \alpha$ and $\mathfrak{N} \not\models_L \alpha$ (this requires the Relativization Property), as well as a *first-order* sentença θ_2 true in $\mathfrak{M}$ saying that $\mathfrak{M} \simeq_p \mathfrak{N}$ via the partial isomorphism I. By the Löwenheim–Skolem Property, θ_1 and θ_2 are jointly true in a enumerável model $\mathfrak{M}_0$ containing partially isomorphic substructures $\mathfrak{M}_0$ and $\mathfrak{N}_0$ such that $\mathfrak{M}_0 \models_L \alpha$ and $\mathfrak{N}_0 \not\models_L \alpha$. But enumerável partially isomorphic estruturas are in fact isomorphic by **??**, contradicting the Isomorphism Property of normal abstract logics. □

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lin.4 Lindström's Theorem mod:lin:prf:sec

Lema lin.8. *Suppose $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, and assume also that there is* mod:lin:prf:lem:lindstrom
an $n \in \mathbb{N}$ such that for any two estruturas $\mathfrak{M}$ and $\mathfrak{N}$, if $\mathfrak{M} \equiv_n \mathfrak{N}$ and $\mathfrak{M} \models_L \alpha$

then also $\mathfrak{N} \models_L \alpha$. Then α is equivalent to a first-order sentença, O.s., there is a first-order θ such that $\mathrm{Mod}_L(\alpha) = \mathrm{Mod}_L(\theta)$.

Prova. Let n be such that any two n-equivalent estruturas $\mathfrak{M}$ and $\mathfrak{N}$ agree on the value assigned to α. Recall **??**: there are only finitely many first-order sentenças in a finite linguagem that have quantifier rank no greater than n, up to logical equivalence. Now, for each fixed estrutura $\mathfrak{M}$ let $\theta_{\mathfrak{M}}$ be the conjunction of all first-order sentenças α true in $\mathfrak{M}$ with $\mathrm{qr}(\alpha) \leq n$ (this conjunction is finite), so that $\mathfrak{N} \models \theta_{\mathfrak{M}}$ if and only if $\mathfrak{N} \equiv_n \mathfrak{M}$. Then put $\theta = \bigvee\{\theta_{\mathfrak{M}} : \mathfrak{M} \models_L \alpha\}$; this disjunction is also finite (up to logical equivalence).

The conclusion $\mathrm{Mod}_L(\alpha) = \mathrm{Mod}_L(\theta)$ follows. In fact, if $\mathfrak{N} \models_L \theta$ then for some $\mathfrak{M} \models_L \alpha$ we have $\mathfrak{N} \models \theta_{\mathfrak{M}}$, whence also $\mathfrak{N} \models_L \alpha$ (by the hypothesis of the lemma). Conversely, if $\mathfrak{N} \models_L \alpha$ then $\theta_{\mathfrak{N}}$ is a disjunct in θ, and since $\mathfrak{N} \models \theta_{\mathfrak{N}}$, also $\mathfrak{N} \models_L \theta$. □

mod:lin:prf:thm:lindstrom **Teorema lin.9 (Lindström's Theorem).** *Suppose $\langle L, \models_L \rangle$ has the Compactness and the Löwenheim–Skolem Properties. Then $\langle L, \models_L \rangle \leq \langle F, \models \rangle$ (so $\langle L, \models_L \rangle$ is equivalent to first-order logic).*

Prova. By Lema lin.8, it suffices to show that for any $\alpha \in L(\mathcal{L})$, with $\mathcal{L}$ finite, there is $n \in \mathbb{N}$ such that for any two estruturas $\mathfrak{M}$ and $\mathfrak{N}$: if $\mathfrak{M} \equiv_n \mathfrak{N}$ then $\mathfrak{M}$ and $\mathfrak{N}$ agree on α. For then α is equivalent to a first-order sentença, from which $\langle L, \models_L \rangle \leq \langle F, \models \rangle$ follows. Since we are working in a finite, purely relational linguagem, by **??** we can replace the statement that $\mathfrak{M} \equiv_n \mathfrak{N}$ by the corresponding algebraic statement that $I_n(\emptyset, \emptyset)$.

Given α, suppose towards a contradiction that for each n there are estruturas $\mathfrak{M}_n$ and $\mathfrak{N}_n$ such that $I_n(\emptyset, \emptyset)$, but (say) $\mathfrak{M}_n \models_L \alpha$ whereas $\mathfrak{N}_n \not\models_L \alpha$. By the Isomorphism Property we can assume that all the $\mathfrak{M}_n$'s interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic sentenças in the language, we may also assume that they satisfy the same atomic sentenças (we can take a subsequence of the $\mathfrak{M}$'s otherwise). Let $\mathfrak{M}$ be the union of all the $\mathfrak{M}_n$'s, O.s., the unique minimal estrutura having each $\mathfrak{M}_n$ as a substructure. As in the proof of Teorema lin.7, let $\mathfrak{M}^*$ be the extension of $\mathfrak{M}$ with domínio $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, in the expanded linguagem comprising the concatenation predicates P and Q.

Similarly, define $\mathfrak{N}_n$, $\mathfrak{N}$ and $\mathfrak{N}^*$. Now let $\mathfrak{M}$ be the estrutura whose domínio comprises the domínios of $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as well as the natural numbers $\mathbb{N}$ along with their natural ordering $\leq$, in the linguagem with extra predicates representing the domínios $|\mathfrak{M}|$, $|\mathfrak{N}|$, $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ as well as predicates coding the domains of $\mathfrak{M}_n$ and $\mathfrak{N}_n$ in the sense that:

$$|\mathfrak{M}_n| = \{a \in |\mathfrak{M}| : R(a,n)\}; \qquad |\mathfrak{N}_n| = \{a \in |\mathfrak{N}| : S(a,n)\};$$
$$|\mathfrak{M}|_n^{<\omega} = \{a \in |\mathfrak{M}|^{<\omega} : R(a,n)\}; \qquad |\mathfrak{N}|_n^{<\omega} = \{a \in |\mathfrak{N}|^{<\omega} : S(a,n)\}.$$

The estrutura $\mathfrak{M}$ also has a ternary relation J such that $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Now there is a sentença θ in the linguagem $\mathcal{L}$ augmented by R, S, J, etc., saying that $\leq$ is a discrete linear ordering with first but no last element and such that $\mathfrak{M}_n \models \alpha$, $\mathfrak{N}_n \not\models \alpha$, and for each n in the ordering, $J(n, \mathbf{a}, \mathbf{b})$ holds if and only if $I_n(\mathbf{a}, \mathbf{b})$.

Using the Compactness Property, we can find a model $\mathfrak{M}^*$ of θ in which the ordering contains a non-standard element n^*. In particular then $\mathfrak{M}^*$ will contain subestruturas $\mathfrak{M}_{n^*}$ and $\mathfrak{N}_{n^*}$ such that $\mathfrak{M}_{n^*} \models_L \alpha$ and $\mathfrak{N}_{n^*} \not\models_L \alpha$. But now we can define a set $\mathcal{I}$ of pairs of k-tuples from $|\mathfrak{M}_{n^*}|$ and $|\mathfrak{N}_{n^*}|$ by putting $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathcal{I}$ if and only if $J(n^* - k, \mathbf{a}, \mathbf{b})$, where k is the length of $\mathbf{a}$ and $\mathbf{b}$. Since n^* is non-standard, for each standard k we have that $n^* - k > 0$, and the set $\mathcal{I}$ witnesses the fact that $\mathfrak{M}_{n^*} \simeq_p \mathfrak{N}_{n^*}$. But by Teorema lin.7, $\mathfrak{M}_{n^*}$ is L-equivalent to $\mathfrak{N}_{n^*}$, a contradiction. □

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Referências Bibliográficas