-NoValue-

## bas.1 Isomorphic Structures

mod:bas:iso: sec

First-order estruturas can be alike in one of two ways. One way in which they can be alike is that they make the same sentenças true. We call such estruturas *elementarily equivalent*. But structures can be very different and still make the same sentenças true—for instance, one can be enumerável and the other not. This is because there are lots of features of a estrutura that cannot be expressed in first-order languages, either because the language is not rich enough, or because of fundamental limitations of first-order logic such as the Löwenheim–Skolem theorem. So another, stricter, aspect in which estruturas can be alike is if they are fundamentally the same, in the sense that they only differ in the objects that make them up, but not in their structural features. A way of making this precise is by the notion of an *isomorphism*.

mod:bas:iso: defn:elem-equiv

**Definição bas.1.** Given two estruturas $\mathfrak{M}$ and $\mathfrak{M}'$ for the same linguagem $\mathcal{L}$, we say that $\mathfrak{M}$ is *elementarily equivalent to* $\mathfrak{M}'$, written $\mathfrak{M} \equiv \mathfrak{M}'$, if and only if for every sentença $\varphi$ of $\mathcal{L}$, $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$.

mod:bas:iso: defn:isomorphism

**Definição bas.2.** Given two estruturas $\mathfrak{M}$ and $\mathfrak{M}'$ for the same linguagem $\mathcal{L}$, we say that $\mathfrak{M}$ is *isomorphic to* $\mathfrak{M}'$, written $\mathfrak{M} \simeq \mathfrak{M}'$, if and only if there is a function $h\colon |\mathfrak{M}| \to |\mathfrak{M}'|$ such that:

1. $h$ is injetiva: if $h(x) = h(y)$ then $x = y$;
2. $h$ is sobrejetiva: for every $y \in |\mathfrak{M}'|$ there is $x \in |\mathfrak{M}|$ such that $h(x) = y$;
3. mod:bas:iso: defn:iso-const — for every símbolo de constante $c$: $h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$;
4. mod:bas:iso: defn:iso-pred — for every $n$-place símbolo de predicado $P$:
$$\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}} \quad \text{iff} \quad \langle h(a_1), \dots, h(a_n) \rangle \in P^{\mathfrak{M}'};$$
5. mod:bas:iso: defn:iso-func — for every $n$-place símbolo de função $f$:
$$h(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{M}'}(h(a_1), \dots, h(a_n)).$$

*mod:bas:iso: thm:isom*

**Teorema bas.3.** *If $\mathfrak{M} \simeq \mathfrak{M}'$ then $\mathfrak{M} \equiv \mathfrak{M}'$.*

*Prova.* Let $h$ be an isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$. For any assignment $s$, $h \circ s$ is the composition of $h$ and $s$, O.s., the assignment in $\mathfrak{M}'$ such that $(h \circ s)(x) = h(s(x))$. By induction on $t$ and $\varphi$ one can prove the stronger claims:

a. $h(\mathrm{Val}^{\mathfrak{M}}_s(t)) = \mathrm{Val}^{\mathfrak{M}'}_{h \circ s}(t)$.

b. $\mathfrak{M}, s \vDash \varphi$ iff $\mathfrak{M}', h \circ s \vDash \varphi$.

The first is proved by induction on the complexity of $t$.

1. If $t \equiv c$, then $\mathrm{Val}_s^{\mathfrak{M}}(c) = c^{\mathfrak{M}}$ and $\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(c) = c^{\mathfrak{M}'}$. Thus, $h(\mathrm{Val}_s^{\mathfrak{M}}(t)) = h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$ (by (3) of Definição bas.2) $= \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t)$.

2. If $t \equiv x$, then $\mathrm{Val}_s^{\mathfrak{M}}(x) = s(x)$ and $\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(x) = h(s(x))$. Thus, $h(\mathrm{Val}_s^{\mathfrak{M}}(x)) = h(s(x)) = \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(x)$.

3. If $t \equiv f(t_1, \dots, t_n)$, then

$$\begin{aligned}
\mathrm{Val}_s^{\mathfrak{M}}(t) &= f^{\mathfrak{M}}(\mathrm{Val}_s^{\mathfrak{M}}(t_1), \dots, \mathrm{Val}_s^{\mathfrak{M}}(t_n)) \quad \text{and} \\
\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t) &= f^{\mathfrak{M}}(\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)).
\end{aligned}$$

   The induction hypothesis is that for each $i$, $h(\mathrm{Val}_s^{\mathfrak{M}}(t_i)) = \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_i)$. So,

$$\begin{aligned}
h(\mathrm{Val}_s^{\mathfrak{M}}(t)) &= h(f^{\mathfrak{M}}(\mathrm{Val}_s^{\mathfrak{M}}(t_1), \dots, \mathrm{Val}_s^{\mathfrak{M}}(t_n)) \\
&= f^{\mathfrak{M}'}(h(\mathrm{Val}_s^{\mathfrak{M}}(t_1)), \dots, h(\mathrm{Val}_s^{\mathfrak{M}}(t_n))) \qquad (1) \\
&= f^{\mathfrak{M}'}(\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)) \qquad (2) \\
&= \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t)
\end{aligned}$$

mod:bas:iso:iso-1
mod:bas:iso:iso-2

   Here, Eq. (1) follows by (5) of Definição bas.2 and Eq. (2) by induction hypothesis.

Part (b) is left as an exercise.

If $\varphi$ is a sentence, the assignments $s$ and $h \circ s$ are irrelevant, and we have $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$. □

**Problema bas.1.** Carry out the proof of (b) of Teorema bas.3 in detail. Make sure to note where each of the five properties characterizing isomorphisms of Definição bas.2 is used.

**Definição bas.4.** An *automorphism* of a structure $\mathfrak{M}$ is an isomorphism of $\mathfrak{M}$ onto itself.

**Problema bas.2.** Show that for any structure $\mathfrak{M}$, if $X$ is a definable subset of $\mathfrak{M}$, and $h$ is an automorphism of $\mathfrak{M}$, then $X = \{h(x) : x \in X\}$ (O.s., $X$ is fixed under $h$).

## Créditos das Imagens

## Referências Bibliográficas