Capítulo udf

Basics of Model Theory

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bas.1 Reducts and Expansions

Often it is useful or necessary to compare languages which have symbols in common, as well as estruturas for these languages. The most common case is when all the symbols in a linguagem $\mathcal{L}$ are also part of a linguagem $\mathcal{L}'$, O.s., $\mathcal{L} \subseteq \mathcal{L}'$. An $\mathcal{L}$-estrutura $\mathfrak{M}$ can then always be expanded to an $\mathcal{L}'$-estrutura by adding interpretations of the additional symbols while leaving the interpretations of the common symbols the same. On the other hand, from an $\mathcal{L}'$-structure $\mathfrak{M}'$ we can obtain an $\mathcal{L}$-structure simply by "forgetting" the interpretations of the symbols that do not occur in $\mathcal{L}$.

mod:bas:red: defn:reduct **Definição bas.1.** Suppose $\mathcal{L} \subseteq \mathcal{L}'$, $\mathfrak{M}$ is an $\mathcal{L}$-estrutura and $\mathfrak{M}'$ is an $\mathcal{L}'$-estrutura. $\mathfrak{M}$ is the *reduct* of $\mathfrak{M}'$ to $\mathcal{L}$, and $\mathfrak{M}'$ is an *expansion* of $\mathfrak{M}$ to $\mathcal{L}'$ iff

1. $|\mathfrak{M}| = |\mathfrak{M}'|$
2. For every símbolo de constante $c \in \mathcal{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$.
3. For every símbolo de função $f \in \mathcal{L}$, $f^{\mathfrak{M}} = f^{\mathfrak{M}'}$.
4. For every símbolo de predicado $P \in \mathcal{L}$, $P^{\mathfrak{M}} = P^{\mathfrak{M}'}$.

mod:bas:red: prop:reduct **Proposição bas.2.** *If an $\mathcal{L}$-estrutura $\mathfrak{M}$ is a reduct of an $\mathcal{L}'$-estrutura $\mathfrak{M}'$, then for all $\mathcal{L}$-sentenças φ,*

$$\mathfrak{M} \vDash \varphi \text{ iff } \mathfrak{M}' \vDash \varphi.$$

Prova. Exercise. □

Problema bas.1. Prove Proposição bas.2.

Definição bas.3. When we have an $\mathcal{L}$-structure $\mathfrak{M}$, and $\mathcal{L}' = \mathcal{L} \cup \{P\}$ is the expansion of $\mathcal{L}$ obtained by adding a single n-place símbolo de predicado P, and $R \subseteq |\mathfrak{M}|^n$ is an n-place relation, then we write $(\mathfrak{M}, R)$ for the expansion $\mathfrak{M}'$ of $\mathfrak{M}$ with $P^{\mathfrak{M}'} = R$.

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bas.2 Subestruturas

The domínio of a estrutura $\mathfrak{M}$ may be a subset of another $\mathfrak{M}'$. But we should obviously only consider $\mathfrak{M}$ a "part" of $\mathfrak{M}'$ if not only $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$, but $\mathfrak{M}$ and $\mathfrak{M}'$ "agree" in how they interpret the symbols of the language at least on the shared part $|\mathfrak{M}|$. mod:bas:sub:sec

Definição bas.4. Given estruturas $\mathfrak{M}$ and $\mathfrak{M}'$ for the same language $\mathcal{L}$, we say that $\mathfrak{M}$ is a *subestrutura* of $\mathfrak{M}'$, and $\mathfrak{M}'$ an *extension* of $\mathfrak{M}$, written $\mathfrak{M} \subseteq \mathfrak{M}'$, iff mod:bas:sub:defn:substructure

1. $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$,
2. For each constant $c \in \mathcal{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$;
3. For each n-place símbolo de função $f \in \mathcal{L}$ $f^{\mathfrak{M}}(a_1, \ldots, a_n) = f^{\mathfrak{M}'}(a_1, \ldots, a_n)$ for all a_1, …, $a_n \in |\mathfrak{M}|$.
4. For each n-place símbolo de predicado $R \in \mathcal{L}$, $\langle a_1, \ldots, a_n \rangle \in R^{\mathfrak{M}}$ iff $\langle a_1, \ldots, a_n \rangle \in R^{\mathfrak{M}'}$ for all a_1, …, $a_n \in |\mathfrak{M}|$.

Observação 1. If the language contains no constant or símbolos de função, then any $N \subseteq |\mathfrak{M}|$ determines a subestrutura $\mathfrak{N}$ of $\mathfrak{M}$ with domínio $|\mathfrak{N}| = N$ by putting $R^{\mathfrak{N}} = R^{\mathfrak{M}} \cap N^n$. mod:bas:sub:rem:substructure

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bas.3 Overspill

mod:bas:ove:sec

Teorema bas.5. *If a set Γ of sentences has arbitrarily large finite models, then it has an infinite model.* *mod:bas:ove:overspill*

Prova. Expand the language of Γ by adding countably many new constants c_0, c_1, … and consider the set $\Gamma \cup \{c_i \neq c_j : i \neq j\}$. To say that Γ has arbitrarily large finite models means that for every $m > 0$ there is $n \geq m$ such that Γ has a model of cardinality n. This implies that $\Gamma \cup \{c_i \neq c_j : i \neq j\}$ is finitely satisfiable. By compactness, $\Gamma \cup \{c_i \neq c_j : i \neq j\}$ has a model $\mathfrak{M}$ whose domain must be infinite, since it satisfies all inequalities $c_i \neq c_j$. □

mod:bas:ove: inf-not-fo **Proposição bas.6.** *There is no sentence φ of any first-order language that is true in a estrutura $\mathfrak{M}$ if and only if the domain $|\mathfrak{M}|$ of the estrutura is infinite.*

Prova. If there were such a φ, its negation $\neg\varphi$ would be true in all and only the finite estruturas, and it would therefore have arbitrarily large finite models but it would lack an infinite model, contradicting Teorema bas.5. □

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bas.4 Isomorphic Structures

mod:bas:iso: sec First-order estruturas can be alike in one of two ways. One way in which they can be alike is that they make the same sentenças true. We call such estruturas *elementarily equivalent*. But structures can be very different and still make the same sentenças true—for instance, one can be enumerável and the other not. This is because there are lots of features of a estrutura that cannot be expressed in first-order languages, either because the language is not rich enough, or because of fundamental limitations of first-order logic such as the Löwenheim–Skolem theorem. So another, stricter, aspect in which estruturas can be alike is if they are fundamentally the same, in the sense that they only differ in the objects that make them up, but not in their structural features. A way of making this precise is by the notion of an *isomorphism*.

mod:bas:iso: defn:elem-equiv **Definição bas.7.** Given two estruturas $\mathfrak{M}$ and $\mathfrak{M}'$ for the same linguagem $\mathcal{L}$, we say that $\mathfrak{M}$ is *elementarily equivalent to* $\mathfrak{M}'$, written $\mathfrak{M} \equiv \mathfrak{M}'$, if and only if for every sentença φ of $\mathcal{L}$, $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$.

mod:bas:iso: defn:isomorphism **Definição bas.8.** Given two estruturas $\mathfrak{M}$ and $\mathfrak{M}'$ for the same linguagem $\mathcal{L}$, we say that $\mathfrak{M}$ is *isomorphic to* $\mathfrak{M}'$, written $\mathfrak{M} \simeq \mathfrak{M}'$, if and only if there is a function $h\colon |\mathfrak{M}| \to |\mathfrak{M}'|$ such that:

1. h is injetiva: if $h(x) = h(y)$ then $x = y$;
2. h is sobrejetiva: for every $y \in |\mathfrak{M}'|$ there is $x \in |\mathfrak{M}|$ such that $h(x) = y$;
3. mod:bas:iso: defn:iso-const for every símbolo de constante c: $h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$;
4. mod:bas:iso: defn:iso-pred for every n-place símbolo de predicado P:
$$\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}} \quad \text{iff} \quad \langle h(a_1), \dots, h(a_n) \rangle \in P^{\mathfrak{M}'};$$
5. mod:bas:iso: defn:iso-func for every n-place símbolo de função f:
$$h(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{M}'}(h(a_1), \dots, h(a_n)).$$

mod:bas:iso: thm:isom **Teorema bas.9.** *If $\mathfrak{M} \simeq \mathfrak{M}'$ then $\mathfrak{M} \equiv \mathfrak{M}'$.*

Prova. Let h be an isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$. For any assignment s, $h \circ s$ is the composition of h and s, O.s., the assignment in $\mathfrak{M}'$ such that $(h \circ s)(x) = h(s(x))$. By induction on t and φ one can prove the stronger claims:

a. $h(\mathrm{Val}_s^{\mathfrak{M}}(t)) = \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t)$.

b. $\mathfrak{M}, s \vDash \varphi$ iff $\mathfrak{M}', h \circ s \vDash \varphi$.

The first is proved by induction on the complexity of t.

1. If $t \equiv c$, then $\mathrm{Val}_s^{\mathfrak{M}}(c) = c^{\mathfrak{M}}$ and $\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(c) = c^{\mathfrak{M}'}$. Thus, $h(\mathrm{Val}_s^{\mathfrak{M}}(t)) = h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$ (by (3) of Definição bas.8) $= \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t)$.

2. If $t \equiv x$, then $\mathrm{Val}_s^{\mathfrak{M}}(x) = s(x)$ and $\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(x) = h(s(x))$. Thus, $h(\mathrm{Val}_s^{\mathfrak{M}}(x)) = h(s(x)) = \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(x)$.

3. If $t \equiv f(t_1, \dots, t_n)$, then

$$\mathrm{Val}_s^{\mathfrak{M}}(t) = f^{\mathfrak{M}}(\mathrm{Val}_s^{\mathfrak{M}}(t_1), \dots, \mathrm{Val}_s^{\mathfrak{M}}(t_n)) \quad \text{and}$$
$$\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t) = f^{\mathfrak{M}}(\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)).$$

   The induction hypothesis is that for each i, $h(\mathrm{Val}_s^{\mathfrak{M}}(t_i)) = \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_i)$. So,

$$\begin{aligned} h(\mathrm{Val}_s^{\mathfrak{M}}(t)) &= h(f^{\mathfrak{M}}(\mathrm{Val}_s^{\mathfrak{M}}(t_1), \dots, \mathrm{Val}_s^{\mathfrak{M}}(t_n)) \\ &= f^{\mathfrak{M}'}(h(\mathrm{Val}_s^{\mathfrak{M}}(t_1)), \dots, h(\mathrm{Val}_s^{\mathfrak{M}}(t_n))) && \text{(bas.1)} \\ &= f^{\mathfrak{M}'}(\mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)) && \text{(bas.2)} \\ &= \mathrm{Val}_{h \circ s}^{\mathfrak{M}'}(t) \end{aligned}$$

mod:bas:iso:iso-1
mod:bas:iso:iso-2

   Here, Eq. (bas.1) follows by (5) of Definição bas.8 and Eq. (bas.2) by induction hypothesis.

Part (b) is left as an exercise.

If φ is a sentence, the assignments s and $h \circ s$ are irrelevant, and we have $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \varphi$. □

Problema bas.2. Carry out the proof of (b) of Teorema bas.9 in detail. Make sure to note where each of the five properties characterizing isomorphisms of Definição bas.8 is used.

Definição bas.10. An *automorphism* of a structure $\mathfrak{M}$ is an isomorphism of $\mathfrak{M}$ onto itself.

Problema bas.3. Show that for any structure $\mathfrak{M}$, if X is a definable subset of $\mathfrak{M}$, and h is an automorphism of $\mathfrak{M}$, then $X = \{h(x) : x \in X\}$ (O.s., X is fixed under h).

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bas.5 The Theory of a Estrutura

Every estrutura $\mathfrak{M}$ makes some sentenças true, and some false. The set of all the sentenças it makes true is called its *theory*. That set is in fact a theory, since anything it entails must be true in all its models, including $\mathfrak{M}$.

Definição bas.11. Given a estrutura $\mathfrak{M}$, the *theory* of $\mathfrak{M}$ is the set $\mathrm{Th}(\mathfrak{M})$ of sentenças that are true in $\mathfrak{M}$, O.s., $\mathrm{Th}(\mathfrak{M}) = \{\varphi : \mathfrak{M} \vDash \varphi\}$.

We also use the term "theory" informally to refer to sets of sentenças having an intended interpretation, whether deductively closed or not.

Proposição bas.12. *For any $\mathfrak{M}$, $\mathrm{Th}(\mathfrak{M})$ is complete.*

Prova. For any sentença φ either $\mathfrak{M} \vDash \varphi$ or $\mathfrak{M} \vDash \neg\varphi$, so either $\varphi \in \mathrm{Th}(\mathfrak{M})$ or $\neg\varphi \in \mathrm{Th}(\mathfrak{M})$. □

mod:bas:thm:prop:equiv

Proposição bas.13. *If $\mathfrak{N} \models \varphi$ for every $\varphi \in \mathrm{Th}(\mathfrak{M})$, then $\mathfrak{M} \equiv \mathfrak{N}$.*

Prova. Since $\mathfrak{N} \vDash \varphi$ for all $\varphi \in \mathrm{Th}(\mathfrak{M})$, $\mathrm{Th}(\mathfrak{M}) \subseteq \mathrm{Th}(\mathfrak{N})$. If $\mathfrak{N} \vDash \varphi$, then $\mathfrak{N} \nvDash \neg\varphi$, so $\neg\varphi \notin \mathrm{Th}(\mathfrak{M})$. Since $\mathrm{Th}(\mathfrak{M})$ is complete, $\varphi \in \mathrm{Th}(\mathfrak{M})$. So, $\mathrm{Th}(\mathfrak{N}) \subseteq \mathrm{Th}(\mathfrak{M})$, and we have $\mathfrak{M} \equiv \mathfrak{N}$. □

mod:bas:thm:remark:R

Observação 2. Consider $\mathfrak{R} = \langle \mathbb{R}, < \rangle$, the estrutura whose domain is the set $\mathbb{R}$ of the real numbers, in the linguagem comprising only a 2-place símbolo de predicado interpreted as the $<$ relation over the reals. Clearly $\mathfrak{R}$ is não enumerável; however, since $\mathrm{Th}(\mathfrak{R})$ is obviously consistent, by the Löwenheim–Skolem theorem it has a enumerável model, say $\mathfrak{S}$, and by Proposição bas.13, $\mathfrak{R} \equiv \mathfrak{S}$. Moreover, since $\mathfrak{R}$ and $\mathfrak{S}$ are not isomorphic, this shows that the converse of Teorema bas.9 fails in general.

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bas.6 Partial Isomorphisms

Definição bas.14. Given two estruturas $\mathfrak{M}$ and $\mathfrak{N}$, a *partial isomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ is a finite partial function p taking arguments in $|\mathfrak{M}|$ and returning values in $|\mathfrak{N}|$, which satisfies the isomorphism conditions from Definição bas.8 on its domain:

1. p is injetiva;
2. for every símbolo de constante c: if $p(c^{\mathfrak{M}})$ is defined, then $p(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$;
3. for every n-place símbolo de predicado P: if $a_1, \ldots, a_n$ are in the domain of p, then $\langle a_1, \ldots, a_n \rangle \in P^{\mathfrak{M}}$ if and only if $\langle p(a_1), \ldots, p(a_n) \rangle \in P^{\mathfrak{N}}$;
4. for every n-place símbolo de função f: if $a_1, \ldots, a_n$ are in the domain of p, then $p(f^{\mathfrak{M}}(a_1, \ldots, a_n)) = f^{\mathfrak{N}}(p(a_1), \ldots, p(a_n))$.

That p is finite means that $\mathrm{dom}(p)$ is finite.

Notice that the empty function $\emptyset$ is always a partial isomorphism between any two estruturas.

Definição bas.15. Two estruturas $\mathfrak{M}$ and $\mathfrak{N}$, are *partially isomorphic*, written $\mathfrak{M} \simeq_p \mathfrak{N}$, if and only if there is a non-empty set I of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ satisfying the *back-and-forth* property: mod:bas:pis:defn:partialisom

1. (*Forth*) For every $p \in I$ and $a \in |\mathfrak{M}|$ there is $q \in I$ such that $p \subseteq q$ and a is in the domain of q;

2. (*Back*) For every $p \in I$ and $b \in |\mathfrak{N}|$ there is $q \in I$ such that $p \subseteq q$ and b is in the range of q.

Teorema bas.16. *If $\mathfrak{M} \simeq_p \mathfrak{N}$ and $\mathfrak{M}$ and $\mathfrak{N}$ are enumerável, then $\mathfrak{M} \simeq \mathfrak{N}$.* *mod:bas:pis:thm:p-isom1*

Prova. Since $\mathfrak{M}$ and $\mathfrak{N}$ are enumerável, let $|\mathfrak{M}| = \{a_0, a_1, \ldots\}$ and $|\mathfrak{N}| = \{b_0, b_1, \ldots\}$. Starting with an arbitrary $p_0 \in I$, we define an increasing sequence of partial isomorphisms $p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots$ as follows:

1. if $n+1$ is odd, say $n = 2r$, then using the Forth property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and a_r is in the domain of p_{n+1};

2. if $n + 1$ is even, say $n + 1 = 2r$, then using the Back property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and b_r is in the range of p_{n+1}.

If we now put:

$$p = \bigcup_{n \geq 0} p_n,$$

we have that p is a an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$. □

Problema bas.4. Show in detail that p as defined in Teorema bas.16 is in fact an isomorphism.

Teorema bas.17. *Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are estruturas for a purely relational linguagem (a linguagem containing only símbolos de predicado, and no símbolos de função or constants). Then if $\mathfrak{M} \simeq_p \mathfrak{N}$, also $\mathfrak{M} \equiv \mathfrak{N}$.* *mod:bas:pis:thm:p-isom2*

Prova. By induction on fórmulas, one shows that if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are such that there is a partial isomorphism p mapping each a_i to b_i and $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ (for $i = 1, \ldots, n$), then $\mathfrak{M}, s_1 \vDash \varphi$ if and only if $\mathfrak{N}, s_2 \vDash \varphi$. The case for $n = 0$ gives $\mathfrak{M} \equiv \mathfrak{N}$. □

Observação 3. If símbolos de função are present, the previous result is still true, but one needs to consider the isomorphism induced by p between the subestrutura of $\mathfrak{M}$ generated by $a_1, \ldots, a_n$ and the subestrutura of $\mathfrak{N}$ generated by $b_1, \ldots, b_n$.

The previous result can be "broken down" into stages by establishing a connection between the number of nested quantifiers in a fórmula and how many times the relevant partial isomorphisms can be extended.

Definição bas.18. For any fórmula φ, the *quantifier rank* of φ, denoted by $\mathrm{qr}(\varphi) \in \mathbb{N}$, is recursively defined as the highest number of nested quantifiers in φ. Two estruturas $\mathfrak{M}$ and $\mathfrak{N}$ are *n-equivalent*, written $\mathfrak{M} \equiv_n \mathfrak{N}$, if they agree on all sentences of quantifier rank less than or equal to n.

mod:bas:pis: prop:qr-finite

Proposição bas.19. *Let $\mathcal{L}$ be a finite purely relational linguagem, O.s., a linguagem containing finitely many símbolos de predicado and símbolos de constante, and no símbolos de função. Then for each $n \in \mathbb{N}$ there are only finitely many first-order sentenças in the linguagem $\mathcal{L}$ that have quantifier rank no greater than n, up to logical equivalence.*

Prova. By induction on n. □

Definição bas.20. Given a estrutura $\mathfrak{M}$, let $|\mathfrak{M}|^{<\omega}$ be the set of all finite sequences over $|\mathfrak{M}|$. We use $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ to range over finite sequences of elements. If $\mathbf{a} \in |\mathfrak{M}|^{<\omega}$ and $a \in |\mathfrak{M}|$, then $\mathbf{a}a$ represents the *concatenation* of $\mathbf{a}$ with a.

Definição bas.21. Given estruturas $\mathfrak{M}$ and $\mathfrak{N}$, we define relations $I_n \subseteq |\mathfrak{M}|^{<\omega} \times |\mathfrak{N}|^{<\omega}$ between sequences of equal length, by recursion on n as follows:

1. $I_0(\mathbf{a}, \mathbf{b})$ if and only if $\mathbf{a}$ and $\mathbf{b}$ satisfy the same atomic fórmulas in $\mathfrak{M}$ and $\mathfrak{N}$; O.s., if $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ and φ is atomic with all variáveis among $x_1, \dots, x_n$, then $\mathfrak{M}, s_1 \vDash \varphi$ if and only if $\mathfrak{N}, s_2 \vDash \varphi$.

2. $I_{n+1}(\mathbf{a}, \mathbf{b})$ if and only if for every $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and vice-versa.

Definição bas.22. Write $\mathfrak{M} \approx_n \mathfrak{N}$ if $I_n(\Lambda, \Lambda)$ holds of $\mathfrak{M}$ and $\mathfrak{N}$ (where Λ is the empty sequence).

mod:bas:pis: thm:b-n-f

Teorema bas.23. *Let $\mathcal{L}$ be a purely relational linguagem. Then $I_n(\mathbf{a}, \mathbf{b})$ implies that for every φ such that $\mathrm{qr}(\varphi) \leq n$, we have $\mathfrak{M}, \mathbf{a} \vDash \varphi$ if and only if $\mathfrak{N}, \mathbf{b} \vDash \varphi$ (where again $\mathbf{a}$ satisfies φ if any s such that $s(x_i) = a_i$ satisfies φ). Moreover, if $\mathcal{L}$ is finite, the converse also holds.*

Prova. The proof that $I_n(\mathbf{a}, \mathbf{b})$ implies that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same fórmulas of quantifier rank no greater than n is by an easy induction on φ. For the converse we proceed by induction on n, using Proposição bas.19, which ensures that for each n there are at most finitely many non-equivalent fórmulas of that quantifier rank.

For $n = 0$ the hypothesis that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same quantifier-free fórmulas gives that they satisfy the same atomic ones, so that $I_0(\mathbf{a}, \mathbf{b})$.

For the $n + 1$ case, suppose that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same fórmulas of quantifier rank no greater than $n + 1$; in order to show that $I_{n+1}(\mathbf{a}, \mathbf{b})$ suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and by the inductive hypothesis again suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same fórmulas of quantifier rank no greater than n.

Given $a \in |\mathfrak{M}|$, let τ_n^a be set of fórmulas $\psi(x, \mathbf{y})$ of rank no greater than n satisfied by $\mathbf{a}a$ in $\mathfrak{M}$; τ_n^a is finite, so we can assume it is a single first-order fórmula. It follows that $\mathbf{a}$ satisfies $\exists x\, \tau_n^a(x, \mathbf{y})$, which has quantifier rank no greater than $n+1$. By hypothesis $\mathbf{b}$ satisfies the same fórmula in $\mathfrak{N}$, so that there is a $b \in |\mathfrak{N}|$ such that $\mathbf{b}b$ satisfies τ_n^a; in particular, $\mathbf{b}b$ satisfies the same fórmulas of quantifier rank no greater than n as $\mathbf{a}a$. Similarly one shows that for every $b \in |\mathfrak{N}|$ there is $a \in |\mathfrak{M}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same fórmulas of quantifier rank no greater than n, which completes the proof. □

Corolário bas.24. *If $\mathfrak{M}$ and $\mathfrak{N}$ are purely relational estruturas in a finite linguagem, then $\mathfrak{M} \approx_n \mathfrak{N}$ if and only if $\mathfrak{M} \equiv_n \mathfrak{N}$. In particular $\mathfrak{M} \equiv \mathfrak{N}$ if and only if for each n, $\mathfrak{M} \approx_n \mathfrak{N}$.* *mod:bas:pis:cor:b-n-f*

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bas.7 Dense Linear Orders

Definição bas.25. A *dense linear ordering without endpoints* is a estrutura $\mathfrak{M}$ for the linguagem containing a single 2-place símbolo de predicado $<$ satisfying the following sentences:

1. $\forall x\, \neg x < x$;
2. $\forall x\, \forall y\, \forall z\, (x < y \rightarrow (y < z \rightarrow x < z))$;
3. $\forall x\, \forall y\, (x < y \lor x = y \lor y < x)$;
4. $\forall x\, \exists y\, x < y$;
5. $\forall x\, \exists y\, y < x$;
6. $\forall x\, \forall y\, (x < y \rightarrow \exists z\, (x < z \land z < y))$.

Teorema bas.26. *Any two enumerável dense linear orderings without endpoints are isomorphic.* *mod:bas:dlo:thm:cantorQ*

Prova. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be enumerável dense linear orderings without endpoints, with $<_1 = <^{\mathfrak{M}_1}$ and $<_2 = <^{\mathfrak{M}_2}$, and let $\mathcal{I}$ be the set of all partial isomorphisms between them. $\mathcal{I}$ is not empty since at least $\emptyset \in \mathcal{I}$. We show that $\mathcal{I}$ satisfies the Back-and-Forth property. Then $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$, and the theorem follows by Teorema bas.16.

To show $\mathcal{I}$ satisfies the Forth property, let $p \in \mathcal{I}$ and let $p(a_i) = b_i$ for $i = 1$, …, n, and without loss of generality suppose $a_1 <_1 a_2 <_1 \cdots <_1 a_n$. Given $a \in |\mathfrak{M}_1|$, find $b \in |\mathfrak{M}_2|$ as follows:

1. if $a <_1 a_1$ let $b \in |\mathfrak{M}_2|$ be such that $b <_2 b_1$;
2. if $a_n <_1 a$ let $b \in |\mathfrak{M}_2|$ be such that $b_n <_2 b$;
3. if $a_i <_1 a <_1 a_{i+1}$ for some i, then let $b \in |\mathfrak{M}_2|$ be such that $b_i <_2 b <_2 b_{i+1}$.

It is always possible to find a b with the desired property since $\mathfrak{M}_2$ is a dense linear ordering without endpoints. Define $q = p \cup \{\langle a, b\rangle\}$ so that $q \in \mathcal{I}$ is the desired extension of p. This establishes the Forth property. The Back property is similar. So $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$; by Teorema bas.16, $\mathfrak{M}_1 \simeq \mathfrak{M}_2$. □

Problema bas.5. Complete the proof of Teorema bas.26 by verifying that $\mathcal{I}$ satisfies the Back property.

Observação 4. Let $\mathfrak{S}$ be any enumerável dense linear ordering without endpoints. Then (by Teorema bas.26) $\mathfrak{S} \simeq \mathfrak{Q}$, where $\mathfrak{Q} = (\mathbb{Q}, <)$ is the enumerável dense linear ordering having the set $\mathbb{Q}$ of the rational numbers as its domain. Now consider again the estrutura $\mathfrak{R} = (\mathbb{R}, <)$ from Observação 2. We saw that there is a enumerável estrutura $\mathfrak{S}$ such that $\mathfrak{R} \equiv \mathfrak{S}$. But $\mathfrak{S}$ is a enumerável dense linear ordering without endpoints, and so it is isomorphic (and hence elementarily equivalent) to the estrutura $\mathfrak{Q}$. By transitivity of elementary equivalence, $\mathfrak{R} \equiv \mathfrak{Q}$. (We could have shown this directly by establishing $\mathfrak{R} \simeq_p \mathfrak{Q}$ by the same back-and-forth argument.)

Créditos das Imagens

Referências Bibliográficas