Capítulo udf

Infinite-valued Logics

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inf.1 Introduction

mvl:inf:int: sec The number of truth values of a matrix need not be finite. An obvious choice for a set of infinitely many truth values is the set of rational numbers between 0 and 1, $V_\infty = [0, 1] \cap \mathbb{Q}$, O.s.,

$$V_\infty = \{\frac{n}{m} : n, m \in \mathbb{N} \text{ and } n \leq m\}.$$

When considering this infinite truth value set, it is often useful to also consider the subsets

$$V_m = \{\frac{n}{m-1} : n \in \mathbb{N} \text{ and } n \leq m\}$$

For instance, V_5 is the set with 5 evenly spaced truth values,

$$V_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}.$$

In logics based on these truth value sets, usually only 1 is designated, O.s., $V^+ = \{1\}$. In other words, we let 1 play the role of (absolute) truth, 0 as absolute falsity, but fórmulas may take any intermediate value in V.

One can also consider the set $V_{[0,1]} = [0, 1]$ of all *real* numbers between 0 and 1, or other infinite subsets of $[0, 1]$, however. Logics with this truth value set are often called *fuzzy*.

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inf.2 Łukasiewicz logic

mvl:inf:luk: sec

This is a short “stub” of a section on infinite-valued Łukasiewicz logic.

Definição inf.1. Infinite-valued Łukasiewicz logic $\mathbf{Ł}_\infty$ is defined using the matrix: mvl:inf:luk: def:lukasiewicz

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\wedge$, $\vee$, $\rightarrow$.
2. The set of truth values V_∞.
3. 1 is the only designated value, O.s., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:

$$\begin{aligned}
\widetilde{\neg}_{\mathbf{Ł}}(x) &= 1 - x \\
\widetilde{\wedge}_{\mathbf{Ł}}(x, y) &= \min(x, y) \\
\widetilde{\vee}_{\mathbf{Ł}}(x, y) &= \max(x, y) \\
\widetilde{\rightarrow}_{\mathbf{Ł}}(x, y) &= \min(1, 1 - (x - y)) = \begin{cases} 1 & \text{if } x \leq y \\ 1 - (x - y) & \text{otherwise.} \end{cases}
\end{aligned}$$

m-valued Łukasiewicz logic is defined the same, except $V = V_m$.

Proposição inf.2. *The logic $\mathbf{Ł}_3$ defined by **??** is the same as $\mathbf{Ł}_3$ defined by Definição inf.1.*

Prova. This can be seen by comparing the truth tables for the connectives given in **??** with the truth tables determined by the equations in Definição inf.1:

$\widetilde{\neg}$	
1	0
1/2	1/2
0	1

$\widetilde{\wedge}_{\mathbf{Ł}_3}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\widetilde{\vee}_{\mathbf{Ł}_3}$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\widetilde{\rightarrow}_{\mathbf{Ł}_3}$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

□

Proposição inf.3. *If $\Gamma \vDash_{\mathbf{Ł}_\infty} \psi$ then $\Gamma \vDash_{\mathbf{Ł}_m} \psi$ for all $m \geq 2$.* *mvl:inf:luk: prop:luk-infty-m*

Prova. Exercise. □

Problema inf.1. Prove Proposição inf.3.

In fact, the converse holds as well.

Infinite-valued Łukasiewicz logic is the most popular fuzzy logic. In the fuzzy logic literature, the conditional is often defined as $\lnot\varphi \lor \psi$. The result would be an infinite-valued strong Kleene logic.

Problema inf.2. Show that $(p \to q) \lor (q \to p)$ is a tautology of $\mathbf{Ł}_\infty$.

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inf.3 Gödel logics

mvl:inf:god:sec

This is a short "stub" of a section on infinite-valued Gödel logic.

mvl:inf:god:def:goedel

Definição inf.4. Infinite-valued Gödel logic $\mathbf{G}_\infty$ is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\bot$, $\lnot$, $\land$, $\lor$, $\to$.
2. The set of truth values V_∞.
3. 1 is the only designated value, O.s., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:

$$\begin{aligned}
\widetilde{\bot} &= 0 \\
\widetilde{\lnot}_{\mathbf{G}}(x) &= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \\
\widetilde{\land}_{\mathbf{G}}(x, y) &= \min(x, y) \\
\widetilde{\lor}_{\mathbf{G}}(x, y) &= \max(x, y) \\
\widetilde{\to}_{\mathbf{G}}(x, y) &= \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise.} \end{cases}
\end{aligned}$$

m-valued Gödel logic is defined the same, except $V = V_m$.

Proposição inf.5. *The logic $\mathbf{G}_3$ defined by **??** is the same as $\mathbf{G}_3$ defined by Definição inf.4.*

Prova. This can be seen by comparing the truth tables for the connectives given in **??** with the truth tables determined by the equations in Definição inf.4:

$\widetilde{\lnot}_{\mathbf{G}_3}$	
1	0
1/2	0
0	1

$\widetilde{\land}_{\mathbf{G}}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\widetilde{\lor}_{\mathbf{G}}$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\widetilde{\to}_{\mathbf{G}}$	1	1/2	0
1	1	1/2	0
1/2	1	1	0
0	1	1	1

□

Proposição inf.6. *If $\Gamma \vDash_{\mathbf{G}_\infty} \psi$ then $\Gamma \vDash_{\mathbf{G}_m} \psi$ for all $m \geq 2$.* mvl:inf:god: prop:god-infty-m

Prova. Exercise. □

Problema inf.3. Prove Proposição inf.6.

In fact, the converse holds as well.

Like $\mathbf{G}_3$, $\mathbf{G}_\infty$ has all intuitionistically valid fórmulas as tautologies, and the same examples of non-tautologies are non-tautologies of $\mathbf{G}_\infty$:

$$
\begin{array}{ll}
p \lor \neg p & (p \to q) \to (\neg p \lor q) \\
\neg\neg p \to p & \neg(\neg p \land \neg q) \to (p \lor q) \\
((p \to q) \to p) \to p & \neg(p \to q) \to (p \land \neg q)
\end{array}
$$

The example of an intuitionistically invalid fórmula that is nevertheless a tautology of $\mathbf{G}_3$, $(p \to q) \lor (q \to p)$, is also a tautology in $\mathbf{G}_\infty$. In fact, $\mathbf{G}_\infty$ can be characterized as intuitionistic logic to which the schema $(\varphi \to \psi) \lor (\psi \to \varphi)$ is added. This was shown by Michael Dummett, and so $\mathbf{G}_\infty$ is often referred to as Gödel–Dummett logic **LC**.

Problema inf.4. Show that $(p \to q) \lor (q \to p)$ is a tautology of $\mathbf{G}_\infty$.

Problema inf.5. Show that $(p \to q) \lor (q \to r) \lor (r \to s)$, which is a tautology of $\mathbf{G}_3$, is not a tautology of $\mathbf{G}_\infty$.

Créditos das Imagens

Referências Bibliográficas