-NoValue-

syn.1 α-Conversion

lam:syn:alp:sec

What is the relation between $\lambda x.\, x$ and $\lambda y.\, y$? They both represent the identity function. They are, of course, syntactically different terms. They differ only in the name of the bound variable, and one is the result of "renaming" the bound variable in the other. This is called *α-conversion*.

Definição syn.1 (Change of bound variable, $\xrightarrow{\alpha}$). If a term M contains an occurrence of $\lambda x.\, N$, $y \notin \mathrm{FV}(N)$, and $N[y/x]$ is defined, then replacing this occurrence by

$$\lambda y.\, N[y/x]$$

resulting in M' is called a *change of bound variable*, written as $M \xrightarrow{\alpha} M'$.

Definição syn.2 (Compatibility of relation). A relation R on terms is said to be *compatible* if it satisfies following conditions:

1. If RNN' then $R\lambda x.\, N\lambda x.\, N'$
2. If RPP' then $R(PQ)(P'Q)$
3. If RQQ' then $R(PQ)(PQ')$

Thus let's rephrase the definition:

Definição syn.3 (Change of bound variable, $\xrightarrow{\alpha}$). *Change of bound variable* ($\xrightarrow{\alpha}$) is the smallest compatible relation on terms satisfying following condition:

$$\lambda x.\, N \xrightarrow{\alpha} \lambda y.\, N[y/x] \qquad \text{if } x \neq y,\ y \notin \mathrm{FV}(N) \text{ and } N[y/x] \text{ is defined}$$

"Smallest" here means the relation contains only pairs that are required by compatibility and the additional condition, and nothing else. Thus this relation can also be defined as follows:

lam:syn:alp:defn:aconvone

Definição syn.4 (Change of bound variable, $\xrightarrow{\alpha}$). *Change of bound variable* ($\xrightarrow{\alpha}$) is inductively defined as follows:

lam:syn:alp:defn:aconvone1
1. If $N \xrightarrow{\alpha} N'$ then $\lambda x.\, N \xrightarrow{\alpha} \lambda x.\, N'$

lam:syn:alp:defn:aconvone2
2. If $P \xrightarrow{\alpha} P'$ then $(PQ) \xrightarrow{\alpha} (P'Q)$

lam:syn:alp:defn:aconvone3
3. If $Q \xrightarrow{\alpha} Q'$ then $(PQ) \xrightarrow{\alpha} (PQ')$

lam:syn:alp:defn:aconvone4
4. If $x \neq y$, $y \notin \mathrm{FV}(N)$ and $N[y/x]$ is defined, then $\lambda x.\, N \xrightarrow{\alpha} \lambda y.\, N[y/x]$.

The definitions are equivalent, but we leave the proof as an exercise. From now on we will use the inductive definition.

Definição syn.5 (α-conversion, $\xrightarrow{\alpha}\!\!\!\!\rightarrow$). *$\alpha$-conversion* ($\xrightarrow{\alpha}\!\!\!\!\rightarrow$) is the smallest reflexive and transitive relation on terms containing $\xrightarrow{\alpha}$.

As above, "smallest" means the relation only contains pairs required by transitivity, and $\xrightarrow{\alpha}$, which leads to the following equivalent definition:

Definição syn.6 (α-conversion, $\xrightarrow{\alpha}\!\!\!\!\rightarrow$). *$\alpha$-conversion* ($\xrightarrow{\alpha}\!\!\!\!\rightarrow$) is inductively defined as follows: lam:syn:alp: defn:aconv

1. If $P \xrightarrow{\alpha}\!\!\!\!\rightarrow Q$ and $Q \xrightarrow{\alpha}\!\!\!\!\rightarrow R$, then $P \xrightarrow{\alpha}\!\!\!\!\rightarrow R$. lam:syn:alp: defn:aconv1
2. If $P \xrightarrow{\alpha} Q$, then $P \xrightarrow{\alpha}\!\!\!\!\rightarrow Q$. lam:syn:alp: defn:aconv2
3. $P \xrightarrow{\alpha}\!\!\!\!\rightarrow P$. lam:syn:alp: defn:aconv3

Exemplo syn.7. $\lambda x.\, fx$ α-converts to $\lambda y.\, fy$, and conversely. Informally speaking, they are both functions that accept an argument and return f of that argument, refering to the environment variable f.

$\lambda x.\, fx$ does not α-convert to $\lambda x.\, gx$. Informally speaking, they refer to the environment variables f and g respectively, and this makes them different functions: they behave differently in environments where f and g are different.

Problema syn.1. Are the following pairs of terms α-convertible?

1. $\lambda x.\, \lambda y.\, x$ and $\lambda y.\, \lambda x.\, y$
2. $\lambda x.\, \lambda y.\, x$ and $\lambda c.\, \lambda b.\, a$
3. $\lambda x.\, \lambda y.\, x$ and $\lambda c.\, \lambda b.\, a$

Lema syn.8. *If $P \xrightarrow{\alpha} Q$ then* $\mathrm{FV}(P) = \mathrm{FV}(Q)$. *lam:syn:alp: lem:fv-one*

Prova. By induction on the derivação of $P \xrightarrow{\alpha} Q$.

1. If the last rule is (4), then P is of the form $\lambda x.\, N$ and Q of the form $\lambda y.\, N[y/x]$, with $x \neq y$, $y \notin \mathrm{FV}(N)$ and $N[y/x]$ defined. We distinguish cases according to whether $x \in \mathrm{FV}(N)$:

   a) If $x \in FV(N)$, then:

$$\begin{aligned}
\mathrm{FV}(\lambda y.\, N[y/x]) &= \mathrm{FV}(N[y/x]) \setminus \{y\} \\
&= ((\mathrm{FV}(N) \setminus \{x\}) \cup \{y\}) \setminus \{y\} && \text{by \textbf{??}} \\
&= \mathrm{FV}(N) \setminus \{x\} \\
&= \mathrm{FV}(\lambda x.\, N)
\end{aligned}$$

b) If $x \notin FV(N)$, then:

$$\begin{aligned}
\mathrm{FV}(\lambda y.\, N[y/x]) &= FV N[y/x] \setminus \{y\} \\
&= \mathrm{FV}(N) \setminus \{x\} && \text{by } \textbf{??} \\
&= \mathrm{FV}(\lambda x.\, N).
\end{aligned}$$

2. The other three cases are left as exercises. □

Problema syn.2. Complete the proof of Lema syn.8.

lam:syn:alp: lem:inv

Lema syn.9. *If $P \xrightarrow{\alpha} Q$ then $Q \xrightarrow{\alpha} P$.*

Prova. Induction on the derivação of $P \xrightarrow{\alpha} Q$.

1. If the last rule is (4), then P is of the form $\lambda x.\, N$ and Q of the form $\lambda y.\, N[y/x]$, where $x \neq y$, $y \notin \mathrm{FV}(N)$ and $N[y/x]$ defined. First, we have $y \notin \mathrm{FV}(N[y/x])$ by **??**. By **??** we have that $N[y/x][x/y]$ is not only defined, but also equal to N. Then by (4), we have $\lambda y.\, N[y/x] \xrightarrow{\alpha} \lambda x.\, N[y/x][x/y] = \lambda x.\, N$. □

Problema syn.3. Complete the proof of Lema syn.9

Teorema syn.10. *α-Conversion is an equivalence relation on terms, O.s., it is reflexive, symmetric, and transitive.*

Prova.
1. For each term M, M can be changed to M by *zero* changes of bound variables.
2. If P is α-converts to Q by a series of changes of bound variables, then from Q we can just inverse these changes (by Lema syn.9) in opposite order to obtain P.
3. If P α-converts to Q by a series of changes of bound variables, and Q to R by another series, then we can change P to R by first applying the first series and then the second series. □

From now on we say that M and N are *α-equivalent*, $M \overset{\alpha}{=} N$, iff M α-converts to N (which, as we've just shown, is the case iff N α-converts to M).

lam:syn:alp: thm:fv

Teorema syn.11. *If $M \overset{\alpha}{=} N$, then* $\mathrm{FV}(M) = \mathrm{FV}(N)$.

Prova. Immediate from Lema syn.8. □

lam:syn:alp: lem:sub:R

Lema syn.12. *If $R \overset{\alpha}{=} R'$ and $M[R/y]$ is defined, then $M[R'/y]$ is defined and α-equivalent to $M[R/y]$.*

Prova. Exercise. □

Problema syn.4. Prove Lema syn.12.

Recall that in **??**, substitution is undefined in some cases; however, using α-conversion on terms, we can make substitution always defined by renaming bound variables. The result preserves α-equivalence, as shown in this theorem:

Teorema syn.13. *For any M, R, and y, there exists M' such that $M \overset{\alpha}{\equiv} M'$ and $M'[R/y]$ is defined. Moreover, if there is another pair $M'' \overset{\alpha}{\equiv} M$ and R'' where $M''[R''/y]$ is defined and $R'' \overset{\alpha}{\equiv} R$, then $M'[R/y] \overset{\alpha}{\equiv} M''[R''/y]$.* lam:syn:alp:thm:sub

Prova. By induction on the formation of M:

1. M is a variable z: Exercise.

2. Suppose M is of the form $\lambda x.\, N$. Select a variable z other than x and y and such that $z \notin \mathrm{FV}(N)$ and $z \notin \mathrm{FV}(R)$. By inductive hypothesis, we there is N' such that $N' \overset{\alpha}{\equiv} N$ and $N'[z/x]$ is defined. Then $\lambda x.\, N \overset{\alpha}{\equiv} \lambda x.\, N'$ too, by Definição syn.4(1). Now $\lambda x.\, N' \overset{\alpha}{\equiv} \lambda z.\, N'[z/x]$ by Definição syn.4(4). We can do this because $z \neq x$, $z \notin FV(N')$ and $N'[z/x]$ is defined. Finally, $\lambda z.\, N'[z/x][R/y]$ is defined, because $z \neq y$ and $z \notin FV(R)$.

   Moreover, if there is another N'' and R'' satisfying the same conditions,

$$\begin{aligned}
(\lambda z.\, N''[z/x])[R''/y] &= \\
&= \lambda z.\, N''[z/x][R''/y] \\
&= \lambda z.\, N''[z/x][R/y] && \text{by Lema syn.12} \\
&= \lambda z.\, N'[z/x][R/y] && \text{by inductive hypothesis} \\
&= (\lambda z.\, N'[z/x])[R/y]
\end{aligned}$$

3. M is of the form (PQ): Exercise. □

Problema syn.5. Complete the proof of Teorema syn.13.

Corolário syn.14. *For any M, R, and y, there exists a pair of M' and R' such that $M' \overset{\alpha}{\equiv} M$, $R \overset{\alpha}{\equiv} R'$ and $M'[R'/y]$ is defined. Moreover, if there is another pair $M'' \overset{\alpha}{\equiv} M$ and R'' with $M'[R'/y]$ defined, then $M'[R'/y] \overset{\alpha}{\equiv} M''[R''/y]$.* lam:syn:alp:cor:sub

Prova. Immediate from Teorema syn.13. □

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Referências Bibliográficas