

Capítulo udf

Intuitionistic Tableaux

Draft chapter on prefixed tableaux for intuitionistic logic. Needs more examples, completeness proofs, and discussion of how one can find countermodels from unsuccessful searches for closed tableaux.

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tab.1 Introduction

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sec **Tableaux** are certain (downward-branching) trees of **fórmulas assinadas**, O.s., pairs consisting of a truth value sign ($\mathbb{T}$ or $\mathbb{F}$) and a **sentença**

$$\mathbb{T} \varphi \text{ or } \mathbb{F} \varphi.$$

A **tableau** begins with a number of *assumptions*. Each further **fórmula assinada** is generated by applying one of the inference rules. Some inference rules add one or more **fórmulas assinadas** to a tip of the tree; others add two new tips, resulting in two branches. Rules result in **fórmulas assinadas** where the **fórmula** is less complex than that of the **fórmula assinada** to which it was applied. When a branch contains both $\mathbb{T} \varphi$ and $\mathbb{F} \varphi$, we say the branch is *closed*. If every branch in a **tableau** is closed, the entire **tableau** is closed. A closed **tableau** constitutes a **derivação** that shows that the set of **fórmulas assinadas** which were used to begin the **tableau** are unsatisfiable. This can be used to define a $\vdash$ relation: $\Gamma \vdash \varphi$ iff there is some finite set $\Gamma_0 = \{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ such that there is a closed **tableau** for the assumptions

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}.$$

For intuitionistic logic, we have to both extend the notion of **fórmula assinada** and adjust the rules for the connectives. In addition to a sign($\mathbb{T}$ or $\mathbb{F}$), **fórmulas** in modal **tableaux** also have *prefixes* σ . The prefixes are non-empty

$\frac{\sigma \mathbb{T} \varphi \wedge \psi}{\sigma \mathbb{T} \varphi} \wedge \mathbb{T}$	$\frac{\sigma \mathbb{F} \varphi \wedge \psi}{\sigma \mathbb{F} \varphi \mid \sigma \mathbb{F} \psi} \wedge \mathbb{F}$
$\frac{\sigma \mathbb{T} \varphi \vee \psi}{\sigma \mathbb{T} \varphi \mid \sigma \mathbb{T} \psi} \vee \mathbb{T}$	$\frac{\sigma \mathbb{F} \varphi \vee \psi}{\sigma \mathbb{F} \varphi} \vee \mathbb{F}$

Tabela tab.1: Prefixed **tableau** rules for $\wedge$ and $\vee$

sequences of positive integers, O.s., $\sigma \in (\mathbb{Z}^+)^* \setminus \{A\}$. When we write such prefixes without the surrounding $\langle \rangle$, and separate the individual **membros** by $\cdot$'s instead of $,$'s. If σ is a prefix, then $\sigma.n$ is $\sigma \frown \langle n \rangle$; p.ex., if $\sigma = 1.2.1$, then $\sigma.3$ is $1.2.1.3$. So for instance,

$$1.2 \mathbb{T} \varphi \rightarrow (\psi \rightarrow \chi)$$

is a *prefixed fórmula assinada* (or just a *prefixed fórmula* for short).

Intuitively, the prefix names a world in a model that might satisfy the **fórmulas** on a branch of a **tableau**, and if σ names some world, then $\sigma.n$ names a world accessible from (the world named by) σ .

In intuitionistic models, the accessibility relation is reflexive and transitive. In terms of prefixes, this means that σ is accessible from σ itself, and so is any prefix that extends σ , O.s., any prefix of the form $\sigma.n_1 \cdots n_k$. Let's introduce the notation $\sigma.*$ to indicate σ itself and any extension of it. In other words, the prefixes $\sigma.*$ are all and only the prefixes accessible from σ .

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tab.2 Rules for Intuitionistic Logic

The rules for the connectives $\wedge$ and $\vee$ are the same as for regular propositional signed **tableaux**, just with prefixes added. In each case, the rule applied to a signed **fórmula** $\sigma S \varphi$ produces new **fórmulas** that are also prefixed by σ . This should be intuitively clear: p.ex., if $\varphi \wedge \psi$ is true at (a world named by) σ , then φ and ψ are true at σ (and not at any other world). We collect the rules for $\wedge$ and $\vee$ in **Tabela tab.1**.

The closure condition is similar to that for ordinary **tableaux**, although we require that not just the **fórmulas**, but also that the prefixes must match. In fact, we can be somewhat more liberal: Since in intuitionistic models, **fórmulas**, once true, remain true, it is impossible that φ is true at σ but false at any accessible prefix $\sigma.*$. So a branch is closed if it contains both

$$\sigma \mathbb{T} \varphi \quad \text{and} \quad \sigma.* \mathbb{F} \varphi$$

$\frac{\sigma \mathbb{T} \neg \varphi}{\sigma.* \mathbb{F} \varphi} \neg \mathbb{T}$ $\sigma.*$ is used	$\frac{\sigma \mathbb{F} \neg \varphi}{\sigma.n \mathbb{T} \varphi} \neg \mathbb{F}$ $\sigma.n$ is new
$\frac{\sigma \mathbb{T} \varphi \rightarrow \psi}{\sigma.* \mathbb{F} \varphi \quad \quad \sigma.* \mathbb{T} \psi} \rightarrow \mathbb{T}$ $\sigma.*$ is used	$\frac{\sigma \mathbb{F} \varphi \rightarrow \psi}{\sigma.n \mathbb{T} \varphi \quad \sigma.n \mathbb{F} \psi} \rightarrow \mathbb{F}$ $\sigma.n$ is new

Tabela tab.2: Prefixed **tableau** rules for $\neg$ and $\rightarrow$

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for some prefix σ and **fórmula** φ . Note that if the signs are reversed, O.s., if it contains

$$\sigma \mathbb{F} \varphi \quad \text{and} \quad \sigma.* \mathbb{T} \varphi$$

the branch is closed only if $*$ is the empty sequence.

In addition, a branch is closed if it contains $\sigma \mathbb{T} \perp$.

The rules for setting up assumptions is also as for ordinary **tableaux**, except that for assumptions we always use the prefix 1. (It does not matter which prefix we use, as long as it's the same for all assumptions.) So, p.ex., we say that

$$\psi_1, \dots, \psi_n \vdash \varphi$$

iff there is a closed tableau for the assumptions

$$1 \mathbb{T} \psi_1, \dots, 1 \mathbb{T} \psi_n, 1 \mathbb{F} \varphi.$$

For the conditional $\rightarrow$, the rules differ from the classical and modal cases. The $\mathbb{T} \rightarrow$ rule extends a branch containing $\sigma \mathbb{T} \varphi \rightarrow \psi$ by $\sigma.* \mathbb{T} \varphi$ and $\sigma.* \mathbb{F} \psi$ on two different branches. It can only be applied for a prefix $\sigma.*$ which *already* occurs on the branch in which it is applied. Let's call such a prefix "used" (on the branch). (Since $\sigma.*$ includes σ itself, the rule can always be applied by adding the prefixed signed formulas $\sigma \mathbb{T} \varphi$ and $\sigma \mathbb{F} \psi$ on separate branches.)

The $\mathbb{F} \rightarrow$ rule extends a branch containing $\sigma \mathbb{F} \varphi \rightarrow \psi$ by both $\sigma.n \mathbb{T} \varphi$ and $\sigma.n \mathbb{F} \psi$ on the same branch, with $\sigma.n$ a prefix new to the branch.

The rules for $\neg$ are defined analogously (using the definition of $\neg \varphi$ as $\varphi \rightarrow \perp$).

The rules are given in **Tabela tab.2**.

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tab.3 Tableaux for Intuitionistic Logic

Exemplo tab.1. We give a closed tableau that shows $(\varphi \wedge \psi) \rightarrow \chi \vdash \varphi \rightarrow (\psi \rightarrow \chi)$.

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1.	$1 \mathbb{T} (\varphi \wedge \psi) \rightarrow \chi$	Assumption
2.	$1 \mathbb{F} \varphi \rightarrow (\psi \rightarrow \chi)$	Assumption
3.	$1.1 \mathbb{T} \varphi$	$\rightarrow \mathbb{F} 2$
4.	$1.1 \mathbb{F} \psi \rightarrow \chi$	$\rightarrow \mathbb{F} 2$
5.	$1.1.1 \mathbb{T} \psi$	$\rightarrow \mathbb{F} 4$
6.	$1.1.1 \mathbb{F} \chi$	$\rightarrow \mathbb{F} 4$
7.	$1.1.1 \mathbb{F} \varphi \wedge \psi$ $1.1.1 \mathbb{T} \chi$	$\rightarrow \mathbb{T} 1$
8.	$1.1.1 \mathbb{F} \varphi$ $1.1.1 \mathbb{F} \psi$	$\wedge \mathbb{F} 4$
	$\otimes$ $\otimes$	

Problema tab.1. Find closed intuitionistic **tableaux** to show the following:

1. $\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$
2. $\vdash \neg(\varphi \wedge \neg\varphi)$
3. $\varphi \rightarrow (\psi \rightarrow \chi) \vdash (\varphi \wedge \psi) \rightarrow \chi$
4. $\neg\varphi \vee \neg\psi \vdash \neg(\varphi \wedge \psi)$

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tab.4 Soundness for Intuitionistic Tableaux

explanation In order to show that intuitionistic **tableaux** are sound, we have to show that if

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$$1 \mathbb{T} \psi_1, \dots, 1 \mathbb{T} \psi_n, 1 \mathbb{F} \varphi$$

has a closed **tableau** then $\psi_1, \dots, \psi_n \models \varphi$. It is easier to prove the contrapositive: if for some $\mathfrak{M}$ and world w , $\mathfrak{M}, w \models \psi_i$ for all $i = 1, \dots, n$ but $\mathfrak{M}, w \not\models \varphi$, then no **tableau** can close. Such a countermodel shows that the initial assumptions of the **tableau** are satisfiable. The strategy of the proof is to show that whenever all the prefixed **fórmulas** on a **tableau** branch are satisfiable, any application of a rule results in at least one extended branch that is also satisfiable. Since closed branches are unsatisfiable, any **tableau** for a satisfiable set of prefixed **fórmulas** must have at least one open branch.

In order to apply this strategy in the modal case, we have to extend our definition of “satisfiable” to relational and prefixes. With that in hand, however, the proof is straightforward.

Definição tab.2. Let P be some set of prefixes, O.s., $P \subseteq (\mathbb{Z}^+)^* \setminus \{A\}$ and let $\mathfrak{M}$ be a model. A function $f: P \rightarrow W$ is an *interpretation of P in $\mathfrak{M}$* if, whenever σ and $\sigma.n$ are both in P , then $Rf(\sigma)f(\sigma.n)$.

Relative to an interpretation of prefixes P we can define:

1. $\mathfrak{M}$ satisfies $\sigma \mathbb{T} \varphi$ iff $\mathfrak{M}, f(\sigma) \Vdash \varphi$.
2. $\mathfrak{M}$ satisfies $\sigma \mathbb{F} \varphi$ iff $\mathfrak{M}, f(\sigma) \nVdash \varphi$.

Note that since R is reflexive and transitive and $\sigma.*$ denotes, $\sigma, \sigma.n_1, \sigma.n_1.n_2, \dots$, we also have that $Rf(\sigma)f(\sigma.*)$.

Definição tab.3. Let Γ be a set of prefixed *fórmulas*, and let $P(\Gamma)$ be the set of prefixes that occur in it. If f is an interpretation of $P(\Gamma)$ in $\mathfrak{M}$, we say that $\mathfrak{M}$ satisfies Γ with respect to f , $\mathfrak{M}, f \Vdash \Gamma$, if $\mathfrak{M}$ satisfies every prefixed *fórmula* in Γ with respect to f . Γ is *satisfiable* iff there is a model $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that $\mathfrak{M}, f \Vdash \Gamma$.

Proposição tab.4. If Γ contains both $\sigma \mathbb{T} \varphi$ and $\sigma.* \mathbb{F} \varphi$ for some *fórmula* φ and prefix σ , or it contains $\sigma \mathbb{T} \perp$, then Γ is unsatisfiable.

Prova. Since always $\mathfrak{M}, f(\sigma) \nVdash \perp$, a Γ that contains $\mathbb{T} \perp$ is unsatisfiable.

There also cannot be a model $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that both If $\mathfrak{M}, f(\sigma) \Vdash \varphi$, then by ??, since $Rf(\sigma)(\sigma.*)$, $\mathfrak{M}, f(\sigma) \Vdash \varphi$. So we cannot have both $\mathfrak{M}, f(\sigma) \Vdash \varphi$ and $\mathfrak{M}, f(\sigma.*) \nVdash \varphi$. $\square$

Teorema tab.5 (Soundness). If Γ has a closed *tableau*, Γ is unsatisfiable.

Prova. We call a branch of a *tableau* satisfiable iff the set of *fórmulas assinadas* on it is satisfiable, and let's call a *tableau* satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable *tableau* by one of the rules of inference always results in a satisfiable *tableau*. This will prove the theorem: any closed *tableau* results by applying rules of inference to the *tableau* consisting only of assumptions from Γ . So if Γ were satisfiable, any *tableau* for it would be satisfiable. A closed *tableau*, however, is clearly not satisfiable, since all its branches are closed and closed branches are unsatisfiable.

Suppose we have a satisfiable *tableau*, O.s., a *tableau* with at least one satisfiable branch. Applying a rule of inference either adds *fórmulas assinadas* to a branch, or splits a branch in two. If the *tableau* has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended *tableau*, so the extended *tableau* is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let Γ be the set of *fórmulas assinadas* on that branch, and let $\sigma S \varphi \in \Gamma$ be the *fórmula assinada* to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, O.s., Γ together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable.

1. The branch is expanded by applying $\neg\mathbb{T}$ to $\sigma\mathbb{T}\neg\psi \in \Gamma$. Then the extended branch contains the **fórmulas assinadas** $\Gamma \cup \{\sigma.*\mathbb{F}\psi\}$. Suppose $\mathfrak{M}, f \Vdash \Gamma$. In particular, $\mathfrak{M}, f(\sigma) \Vdash \neg\psi$. Thus, $\mathfrak{M}, w \not\Vdash \psi$ for any w such that $Rf(\sigma)w$, and that includes $f(\sigma.*)$. So, $\mathfrak{M}$ satisfies $\sigma.*\mathbb{F}\psi$ with respect to f .
2. The branch is expanded by applying $\neg\mathbb{F}$ to $\sigma\mathbb{F}\neg\psi \in \Gamma$: Exercise.
3. The branch is expanded by applying $\wedge\mathbb{T}$ to $\sigma\mathbb{T}\psi \wedge \chi \in \Gamma$, which results in two new **fórmulas assinadas** on the branch: $\sigma\mathbb{T}\psi$ and $\sigma\mathbb{T}\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \Vdash \psi \wedge \chi$. Then $\mathfrak{M}, f(\sigma) \Vdash \psi$ and $\mathfrak{M}, f(\sigma) \Vdash \chi$. This means that $\mathfrak{M}$ satisfies both $\sigma\mathbb{T}\psi$ and $\sigma\mathbb{T}\chi$ with respect to f .
4. The branch is expanded by applying $\vee\mathbb{F}$ to $\mathbb{F}\psi \vee \chi \in \Gamma$: Exercise.
5. The branch is expanded by applying $\rightarrow\mathbb{F}$ to $\sigma\mathbb{F}\psi \rightarrow \chi \in \Gamma$: This results in two new **fórmulas assinadas** on the branch: $\sigma.n\mathbb{T}\psi$ and $\sigma.n\mathbb{F}\chi$, where $\sigma.n$ is a new prefix on the branch, O.s., $\sigma.n \notin P(\Gamma)$.

Since Γ is satisfiable, there is a $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \not\Vdash \psi \rightarrow \chi$. We have to show that $\Gamma \cup \{\sigma.n\mathbb{F}\psi \rightarrow \chi\}$ is satisfiable. To do this, we define an interpretation of $P(\Gamma) \cup \{\sigma.n\}$ as follows:

Since $\mathfrak{M}, f(\sigma) \not\Vdash \psi \rightarrow \chi$, there is a $w \in W$ such that $Rf(\sigma)w$ such that $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \not\Vdash \chi$. Let f' be like f , except that $f'(\sigma.n) = w$. Since $f'(\sigma) = f(\sigma)$ and $Rf(\sigma)w$, we have $Rf'(\sigma)f'(\sigma.n)$, so f' is an interpretation of $P(\Gamma) \cup \{\sigma.n\}$. Obviously $\mathfrak{M}, f'(\sigma.n) \Vdash \psi$ and $\mathfrak{M}, f'(\sigma.n) \not\Vdash \chi$. Since $f(\sigma') = f'(\sigma')$ for all prefixes $\sigma' \in P(\Gamma)$, $\mathfrak{M}, f' \Vdash \Gamma$. So, $\mathfrak{M}, f'$ satisfies $\Gamma \cup \{\sigma.n\mathbb{F}\psi \rightarrow \chi\}$

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying $\wedge\mathbb{F}$ to $\sigma\mathbb{F}\psi \wedge \chi \in \Gamma$, which results in two branches, a left one continuing through $\sigma\mathbb{F}\psi$ and a right one through $\sigma\mathbb{F}\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \not\Vdash \psi \wedge \chi$. Then $\mathfrak{M}, f(\sigma) \not\Vdash \psi$ or $\mathfrak{M}, f(\sigma) \not\Vdash \chi$. In the former case, $\mathfrak{M}, f$ satisfies $\sigma\mathbb{F}\psi$, O.s., the left branch is satisfiable. In the latter, $\mathfrak{M}, f$ satisfies $\sigma\mathbb{F}\chi$, O.s., the right branch is satisfiable.
2. The branch is expanded by applying $\vee\mathbb{T}$ to $\sigma\mathbb{T}\psi \vee \chi \in \Gamma$: Exercise.
3. The branch is expanded by applying $\rightarrow\mathbb{T}$ to $\sigma\mathbb{T}\psi \rightarrow \chi \in \Gamma$: Exercise. $\square$

Problema tab.2. Complete the proof of **Teorema tab.5**.

Corolário tab.6. If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

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cor:entailment-soundness*

Prova. If $\Gamma \vdash \varphi$ then for some $\psi_1, \dots, \psi_n \in \Gamma$, $\Delta = \{1 \mathbb{F} \varphi, 1 \mathbb{T} \psi_1, \dots, 1 \mathbb{T} \psi_n\}$ has a closed **tableau**. We want to show that $\Gamma \models \varphi$. Suppose not, so for some $\mathfrak{M}$ and w , $\mathfrak{M}, w \Vdash \psi_i$ for $i = 1, \dots, n$, but $\mathfrak{M}, w \not\models \varphi$. Let $f(1) = w$; then f is an interpretation of $P(\Delta)$ into $\mathfrak{M}$, and $\mathfrak{M}$ satisfies Δ with respect to f . But by **Teorema tab.5**, Δ is unsatisfiable since it has a closed **tableau**, a contradiction. So we must have $\Gamma \vdash \varphi$ after all. $\square$

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cor:weak-soundness **Corolário tab.7.** *If $\vdash \varphi$ then φ is true in all models.*

Créditos das Imagens

Referências Bibliográficas