Capítulo udf

Soundness and Completeness

This chapter collects soundness and completeness results for propositional intuitionistic logic. It needs an introduction. The completeness proof makes use of facts about provability that should be stated and proved explicitly somewhere.

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sc.1 Soundness of Axiomatic Derivações

int:sc:sax:sec

The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, p.ex., in the Semantics chapter.

int:sc:sax:thm:soundness **Teorema sc.1 (Soundness).** *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*

Prova. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the number n of fórmulas in the derivação of φ from Γ. We show that if φ_1, …, $\varphi_n = \varphi$ is a derivação from Γ, then $\Gamma \vDash \varphi_n$. Note that if φ_1, …, φ_n is a derivação, so is φ_1, …, φ_k for any $k < n$.

There are no derivações of length 0, so for $n = 0$ the claim holds vacuously. So the claim holds for all derivações of length $< n$. We distinguish cases according to the justification of φ_n.

1. φ_n is an axiom. All axioms are valid, so $\Gamma \vDash \varphi_n$ for any Γ.

2. $\varphi_n \in \Gamma$. Then for any $\mathfrak{M}$ and w, if $\mathfrak{M}, w \Vdash \Gamma$, obviously $\mathfrak{M} \Vdash \Gamma\varphi_n[w]$, O.s., $\Gamma \vDash \varphi$.

3. φ_n follows by MP from φ_i and $\varphi_j \equiv \varphi_i \to \varphi_n$. φ_1, …, φ_i and φ_1, …, φ_j are derivações from Γ, so by inductive hypothesis, $\Gamma \vDash \varphi_i$ and $\Gamma \vDash \varphi_i \to \varphi_n$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \varphi_i \to \varphi_n$, $\mathfrak{M}, w \Vdash \varphi_i \to \varphi_n$. By definition, this means that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \varphi_i$ then $\mathfrak{M}, w' \Vdash \varphi_n$. Since R is reflexive, w is among the w' such that Rww', O.s., we have that if $\mathfrak{M}, w \Vdash \varphi_i$ then $\mathfrak{M}, w \Vdash \varphi_n$. Since $\Gamma \vDash \varphi_i$, $\mathfrak{M}, w \Vdash \varphi_i$. So, $\mathfrak{M}, w \Vdash \varphi_n$, as we wanted to show. □

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sc.2 Soundness of Natural Deduction

We will now prove soundness of natural deduction with regards to the relational semantics, that is, showing that if a fórmula is derivável from a set of assumptions then the set of assumptions entails the fórmula. int:sc:snd:sec

Teorema sc.2 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.* *int:sc:snd:thm:soundness*

Prova. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the derivação of φ from Γ.

1. If the derivação consists of just the assumption φ, we have $\varphi \vdash \varphi$, and want to show that $\varphi \vDash \varphi$. Suppose that $\mathfrak{M}, w \Vdash \varphi$. Then trivially $\mathfrak{M}, w \Vdash \varphi$.

2. The derivação ends in $\wedge$Intro: The derivações of the premises ψ from não descarregada assumptions Γ and of χ from não descarregada assumptions Δ show that $\Gamma \vdash \psi$ and $\Delta \vdash \chi$. By induction hypothesis we have that $\Gamma \vDash \psi$ and $\Delta \vDash \chi$. We have to show that $\Gamma \cup \Delta \vDash \varphi \wedge \psi$, since the não descarregada assumptions of the entire derivação are Γ together with Δ. So suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$. Then also $\mathfrak{M}, w \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. Similarly, $\mathfrak{M}, w \Vdash \chi$. So $\mathfrak{M}, w \Vdash \psi \wedge \chi$.

3. The derivação ends in $\wedge$Elim: The derivação of the premise $\psi \wedge \chi$ from não descarregada assumptions Γ shows that $\Gamma \vdash \psi \wedge \chi$. By induction hypothesis, $\Gamma \vDash \psi \wedge \chi$. We have to show that $\Gamma \vDash \psi$. So suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\Gamma \vDash \psi \wedge \chi$, $\mathfrak{M}, w \Vdash \psi \wedge \chi$. Then also $\mathfrak{M}, w \Vdash \psi$. Similarly if $\wedge$Elim ends in χ, then $\Gamma \vDash \chi$.

4. The derivação ends in $\vee$Intro: Suppose the premise is ψ, and the não descarregada assumptions of the derivação ending in ψ are Γ. Then we have $\Gamma \vdash \psi$ and by inductive hypothesis, $\Gamma \vDash \psi$. We have to show that $\Gamma \vDash \psi \vee \chi$. Suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. But then also $\mathfrak{M}, w \Vdash \psi \vee \chi$. Similarly, if the premise is χ, we have that $\Gamma \vDash \chi$.

5. The derivação ends in $\vee$Elim: The derivações ending in the premises are of $\psi \vee \chi$ from não descarregada assumptions Γ, of θ from não descarregada assumptions $\Delta_1 \cup \{\psi\}$, and of θ from não descarregada assumptions $\Delta_2 \cup \{\chi\}$. So we have $\Gamma \vdash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vdash \theta$, and $\Delta_2 \cup \{\chi\} \vdash \theta$. By induction hypothesis, $\Gamma \vDash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vDash \theta$, and $\Delta_2 \cup \{\chi\} \vDash \theta$. We have to prove that $\Gamma \cup \Delta_1 \cup \Delta_2 \vDash \theta$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta_1 \cup \Delta_2$. Then $\mathfrak{M}, w \Vdash \Gamma$ and since $\Gamma \vDash \psi \vee \chi$, $\mathfrak{M}, w \Vdash \psi \vee \chi$. By definition of $\mathfrak{M} \Vdash$, either $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$. So we distinguish cases: (a) $\mathfrak{M} \Vdash \psi[\mathrm{w}]$. Then $\mathfrak{M}, w \Vdash \Delta_1 \cup \{\psi\}$. Since $\Delta_1 \cup \psi \vDash \theta$, we have $\mathfrak{M}, w \Vdash \theta$. (b) $\mathfrak{M}, w \Vdash \chi$. Then $\mathfrak{M}, w \Vdash \Delta_2 \cup \{\chi\}$. Since $\Delta_2 \cup \chi \vDash \theta$, we have $\mathfrak{M}, w \Vdash \theta$. So in either case, $\mathfrak{M}, w \Vdash \theta$, as we wanted to show.

6. The derivação ends with $\to$Intro concluding $\psi \to \chi$. Then the premise is χ, and the derivação ending in the premise has não descarregada assumptions $\Gamma \cup \{\psi\}$. So we have that $\Gamma \cup \{\psi\} \vdash \chi$, and by induction hypothesis that $\Gamma \cup \{\psi\} \vDash \chi$. We have to show that $\Gamma \vDash \psi \to \chi$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma$. We want to show that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \psi$, then $\mathfrak{M}, w' \Vdash \chi$. So assume that Rww' and $\mathfrak{M}, w' \Vdash \psi$. By **??**, $\mathfrak{M}, w' \Vdash \Gamma$. Since $\Gamma \cup \{\psi\} \vDash \chi$, $\mathfrak{M}, w' \Vdash \chi$, which is what we wanted to show.

7. The derivação ends in $\to$Elim and conclusion χ. The premises are $\psi \to \chi$ and ψ, with derivações from não descarregada assumptions Γ, Δ. So we have $\Gamma \vdash \psi \to \chi$ and $\Delta \vdash \psi$. By inductive hypothesis, $\Gamma \vDash \psi \to \chi$ and $\Delta \vDash \psi$. We have to show that $\Gamma \cup \Delta \vDash \chi$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \psi \to \chi$, $\mathfrak{M}, w \Vdash \psi \to \chi$. By definition, this means that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \psi$ then $\mathfrak{M}, w' \Vdash \chi$. Since R is reflexive, w is among the w' such that Rww', O.s., we have that if $\mathfrak{M}, w \Vdash \psi$ then $\mathfrak{M}, w \Vdash \chi$. Since $\mathfrak{M}, w \Vdash \Delta$ and $\Delta \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. So, $\mathfrak{M}, w \Vdash \chi$, as we wanted to show.

8. The derivação ends in $\bot_I$, concluding φ. The premise is $\bot$ and the não descarregada assumptions of the derivação of the premise are Γ. Then $\Gamma \vdash \bot$. By inductive hypothesis, $\Gamma \vDash \bot$. We have to show $\Gamma \vDash \varphi$.

   We proceed indirectly. If $\Gamma \nvDash \varphi$ there is a model $\mathfrak{M}$ and world w such that $\mathfrak{M}, w \Vdash \Gamma$ and $\mathfrak{M}, w \nVdash \varphi$. Since $\Gamma \vDash \bot$, $\mathfrak{M}, w \Vdash \bot$. But that's impossible, since by definition, $\mathfrak{M}, w \nVdash \bot$. So $\Gamma \vDash \varphi$.

9. The derivação ends in $\neg$Intro: Exercise.

10. The derivação ends in $\neg$Elim: Exercise. □

Problema sc.1. Complete the proof of Teorema sc.2. For the cases for $\neg$Intro and $\neg$Elim, use the definition of $\mathfrak{M}, w \Vdash \neg\varphi$ in **??**, O.s., don't treat $\neg\varphi$ as defined by $\varphi \to \bot$.

Problema sc.2. Show that the following fórmulas are not derivável in intuitionistic logic:

1. $(\varphi \to \psi) \lor (\psi \to \varphi)$
2. $(\neg\neg\varphi \to \varphi) \to (\varphi \lor \neg\varphi)$
3. $(\varphi \to \psi \lor \chi) \to \big((\varphi \to \psi) \lor (\varphi \to \chi)\big)$

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sc.3 Lindenbaum's Lemma

The completeness theorem for intuitionistic logic is proved by assuming $\Gamma \nvdash \varphi$ and constructing a model $\mathfrak{M} \Vdash \Gamma$ and $\mathfrak{M} \nVdash \varphi$. int:sc:lin:sec

In classical logic the relation of derivabilidade can be reduced to the notion of consistency since a fórmula φ is derivável from a set of fórmulas iff the set together with the negation of φ is inconsistent. This is not possible in intuitionistic logic. In intuitionistic logic, if $\neg\varphi$ is inconsistent, we only get that $\vdash \neg\neg\varphi$. Since $\neg\neg\varphi \to \varphi$ does not hold intuitionistically in general, we cannot conclude that $\vdash \varphi$.

Thus, when constructing the model $\mathfrak{M}$, we will need to keep track of the non-derivabilidade of the fórmula φ and thus we will not be able to use a complete set $\Gamma^* \supseteq \Gamma$ to build the model $\mathfrak{M}$, as in every complete set Γ^*, we have $\Gamma^* \vdash \varphi \lor \neg\varphi$.

Instead of using a complete set Γ^*, we will us the notion of a prime set of formulas:

Definição sc.3. A set of fórmulas Γ is *prime* iff int:sc:lin:defn:prime

1. Γ is consistent, O.s., $\Gamma \nvdash \bot$; int:sc:lin:defn:prime1
2. if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$; and int:sc:lin:defn:prime2
3. if $\varphi \lor \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$. int:sc:lin:defn:prime3

Lema sc.4 (Lindenbaum's Lemma). *If $\Gamma \nvdash \varphi$, there is a $\Gamma^* \supseteq \Gamma$ such that Γ^* is prime and $\Gamma^* \nvdash \varphi$.* *int:sc:lin:lem:lindenbaum*

Prova. Let $\psi_1 \lor \chi_1$, $\psi_2 \lor \chi_2$, …, be an enumeration of all fórmulas of the form $\psi \lor \chi$. We'll define an increasing sequence of sets of fórmulas Γ_n, where each Γ_{n+1} is defined as Γ_n together with one new fórmula. Γ^* will be the union of all Γ_n. The new fórmulas are selected so as to ensure that Γ^* is prime and still $\Gamma^* \nvdash \varphi$. This means that at each step we should find the first disjunction $\psi_i \lor \chi_i$ such that:

1. $\Gamma_n \vdash \psi_i \lor \chi_i$ int:sc:lin:gamma-1
2. $\psi_i \notin \Gamma_n$ and $\chi_i \notin \Gamma_n$ int:sc:lin:gamma-2

We add to Γ_n either ψ_i if $\Gamma_n \cup \{\psi_i\} \nvdash \varphi$, or χ_i otherwise. We'll have to show that this works. For now, let's define $i(n)$ as the least i such that (1) and (2) hold.

Define $\Gamma_0 = \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_{i(n)}\} & \text{if } \Gamma_n \cup \{\psi_{i(n)}\} \nvdash \varphi \\ \Gamma_n \cup \{\chi_{i(n)}\} & \text{otherwise} \end{cases}$$

If $i(n)$ is undefined, O.s., whenever $\Gamma_n \vdash \psi \lor \chi$, either $\psi \in \Gamma_n$ or $\chi \in \Gamma_n$, we let $\Gamma_{n+1} = \Gamma_n$. Now let $\Gamma^* = \bigcup_{n=0}^{\infty} \Gamma_n$

First we show that for all n, $\Gamma_n \nvdash \varphi$. We proceed by induction on n. For $n = 0$ the claim holds by the hypothesis of the theorem, O.s., $\Gamma \nvdash \varphi$. If $n > 0$, we have to show that if $\Gamma_n \nvdash \varphi$ then $\Gamma_{n+1} \nvdash \varphi$. If $i(n)$ is undefined, $\Gamma_{n+1} = \Gamma_n$ and there is nothing to prove. So suppose $i(n)$ is defined. For simplicity, let $i = i(n)$.

We'll prove the contrapositive of the claim. Suppose $\Gamma_{n+1} \vdash \varphi$. By construction, $\Gamma_{n+1} = \Gamma_n \cup \{\psi_i\}$ if $\Gamma_n \cup \{\psi_i\} \nvdash \varphi$, or else $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. It clearly can't be the first, since then $\Gamma_{n+1} \nvdash \varphi$. Hence, $\Gamma_n \cup \{\psi_i\} \vdash \varphi$ and $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. By definition of $i(n)$, we have that $\Gamma_n \vdash \psi_i \lor \chi_i$. We have $\Gamma_n \cup \{\psi_i\} \vdash \varphi$. We also have $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\} \vdash \varphi$. Hence, $\Gamma_n \vdash \varphi$, which is what we wanted to show.

If $\Gamma^* \vdash \varphi$, there would be some finite subset $\Gamma' \subseteq \Gamma^*$ such that $\Gamma' \vdash \varphi$. Each $\theta \in \Gamma'$ must be in Γ_i for some i. Let n be the largest of these. Since $\Gamma_i \subseteq \Gamma_n$ if $i \leq n$, $\Gamma' \subseteq \Gamma_n$. But then $\Gamma_n \vdash \varphi$, contrary to our proof above that $\Gamma_n \nvdash \varphi$.

Lastly, we show that Γ^* is prime, O.s., satisfies conditions (1), (2), and (3) of Definição sc.3.

First, $\Gamma^* \nvdash \varphi$, so Γ^* is consistent, so (1) holds.

We now show that if $\Gamma^* \vdash \psi \lor \chi$, then either $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$. This proves (3), since if $\psi \lor \chi \in \Gamma^*$ then also $\Gamma^* \vdash \psi \lor \chi$. So assume $\Gamma^* \vdash \psi \lor \chi$ but $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$. Since $\Gamma^* \vdash \psi \lor \chi$, $\Gamma_n \vdash \psi \lor \chi$ for some n. $\psi \lor \chi$ appears on the enumeration of all disjunctions, say, as $\psi_j \lor \chi_j$. $\psi_j \lor \chi_j$ satisfies the properties in the definition of $i(n)$, namely we have $\Gamma_n \vdash \psi_j \lor \chi_j$, while $\psi_j \notin \Gamma_n$ and $\chi_j \notin \Gamma_n$. At each stage, at least one fewer disjunction $\psi_i \lor \chi_i$ satisfies the conditions (since at each stage we add either ψ_i or χ_i), so at some stage m we will have $j = i(m)$. But then either $\psi \in \Gamma_{m+1}$ or $\chi \in \Gamma_{m+1}$, contrary to the assumption that $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$.

Now suppose $\Gamma^* \vdash \psi$. Then $\Gamma^* \vdash \psi \lor \psi$. But we've just proved that if $\Gamma^* \vdash \psi \lor \psi$ then $\psi \in \Gamma^*$. Hence, Γ^* satisfies (2) of Definição sc.3. □

Problema sc.3. Show that if $\Gamma \nvdash \bot$ then Γ is consistent in classical logic, O.s., there is a valuation making all formulas in Γ true.

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sc.4 The Canonical Model

The worlds in our model will be finite sequences σ of natural numbers, O.s., $\sigma \in \mathbb{N}^*$. Note that $\mathbb{N}^*$ is inductively defined by: int:sc:mod:sec

1. $\Lambda \in \mathbb{N}^*$.
2. If $\sigma \in \mathbb{N}^*$ and $n \in \mathbb{N}$, then $\sigma.n \in \mathbb{N}^*$ (where $\sigma.n$ is $\sigma \frown \langle n \rangle$ and $\sigma \frown \sigma'$ is the concatenation if σ and σ').
3. Nothing else is in $\mathbb{N}^*$.

So we can use $\mathbb{N}^*$ to give inductive definitions.

Let $\langle \psi_1, \chi_1 \rangle$, $\langle \psi_2, \chi_2 \rangle$, ..., be an enumeration of all pairs of fórmulas. Given a set of fórmulas Δ, define $\Delta(\sigma)$ by induction as follows:

1. $\Delta(\Lambda) = \Delta$
2. $\Delta(\sigma.n) =$

$$\begin{cases} (\Delta(\sigma) \cup \{\psi_n\})^* & \text{if } \Delta(\sigma) \cup \{\psi_n\} \nvdash \chi_n \\ \Delta(\sigma) & \text{otherwise} \end{cases}$$

Here by $(\Delta(\sigma) \cup \{\psi_n\})^*$ we mean the prime set of fórmulas which exists by Lema sc.4 applied to the set $\Delta(\sigma) \cup \{\psi_n\}$ and the fórmula χ_n. Note that by this definition, if $\Delta(\sigma) \cup \{\psi_n\} \nvdash \chi_n$, then $\Delta(\sigma.n) \vdash \psi_n$ and $\Delta(\sigma.n) \nvdash \chi_n$. Note also that $\Delta(\sigma) \subseteq \Delta(\sigma.n)$ for any n. If Δ is prime, then $\Delta(\sigma)$ is prime for all σ.

Definição sc.5. Suppose Δ is prime. Then the *canonical model* $\mathfrak{M}(\Delta)$ for Δ is defined by: int:sc:mod:defn:canonical-model

1. $W = \mathbb{N}^*$, the set of finite sequences of natural numbers.
2. R is the partial order according to which $R\sigma\sigma'$ iff σ is an initial segment of σ' (O.s., $\sigma' = \sigma \frown \sigma''$ for some sequence σ'').
3. $V(p) = \{\sigma : p \in \Delta(\sigma)\}$.

It is easy to verify that R is indeed a partial order. Also, the monotonicity condition on V is satisfied. Since $\Delta(\sigma) \subseteq \Delta(\sigma.n)$ we get $\Delta(\sigma) \subseteq \Delta(\sigma')$ whenever $R\sigma\sigma'$ by induction on σ.

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sc.5 The Truth Lemma

The following lemma connects satisfaction in the canonical model with which fórmulas are membros of the prime set Δ. int:sc:tru:sec

Lema sc.6. *If Δ is prime, then $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\Delta(\sigma) \vdash \varphi$.* *int:sc:tru:lem:truth*

Prova. By induction on φ.

1. $\varphi \equiv \bot$: Since $\Delta(\sigma)$ is prime, it is consistent, so $\Delta(\sigma) \nvdash \varphi$. By definition, $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

2. $\varphi \equiv p$: By definition of $\Vdash$, $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\sigma \in V(p)$, O.s., $\Delta(\sigma) \vdash \varphi$.

3. $\varphi \equiv \neg\psi$: exercise.

4. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ iff $\Delta(\sigma) \vdash \psi$, and similarly for χ. But $\Delta(\sigma) \vdash \psi$ and $\Delta(\sigma) \vdash \chi$ iff $\Delta(\sigma) \vdash \varphi$.

5. $\varphi \equiv \psi \vee \chi$: $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. By induction hypothesis, this holds iff $\Delta(\sigma) \vdash \psi$ or $\Delta(\sigma) \vdash \chi$. We have to show that this in turn holds iff $\Delta(\sigma) \vdash \varphi$. The left-to-right direction is clear. The right-to-left direction follows since $\Delta(\sigma)$ is prime.

6. $\varphi \equiv \psi \rightarrow \chi$: First the contrapositive of the left-to-right direction: Assume $\Delta(\sigma) \nvdash \psi \rightarrow \chi$. Then also $\Delta(\sigma) \cup \{\psi\} \nvdash \chi$. Since $\langle \psi, \chi \rangle$ is $\langle \psi_n, \chi_n \rangle$ for some n, we have $\Delta(\sigma.n) = (\Delta(\sigma) \cup \{\psi\})^*$, and $\Delta(\sigma.n) \vdash \psi$ but $\Delta(\sigma.n) \nvdash \chi$. By inductive hypothesis, $\mathfrak{M}(\Delta), \sigma.n \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma.n \nVdash \chi$. Since $R\sigma(\sigma.n)$, this means that $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

   Now assume $\Delta(\sigma) \vdash \psi \rightarrow \chi$, and let $R\sigma\sigma'$. Since $\Delta(\sigma) \subseteq \Delta(\sigma')$, we have: if $\Delta(\sigma') \vdash \psi$, then $\Delta(\sigma') \vdash \chi$. In other words, for every σ' such that $R\sigma\sigma'$, either $\Delta(\sigma') \nvdash \psi$ or $\Delta(\sigma') \vdash \chi$. By induction hypothesis, this means that whenever $R\sigma\sigma'$, either $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma' \Vdash \chi$, O.s., $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$. □

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sc.6 The Completeness Theorem

int:sc:cpl: sec

int:sc:cpl: thm:completeness **Teorema sc.7.** *If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.*

Prova. We prove the contrapositive: Suppose $\Gamma \nvdash \varphi$. Then by Lema sc.4, there is a prime set $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \nvdash \varphi$. Consider the canonical model $\mathfrak{M}(\Gamma^*)$ for Γ^* as defined in Definição sc.5. For any $\psi \in \Gamma$, $\Gamma^* \vdash \psi$. Note that $\Gamma^*(\Lambda) = \Gamma^*$. By the Truth Lemma (Lema sc.6), we have $\mathfrak{M}(\Gamma^*), \Lambda \Vdash \psi$ for all $\psi \in \Gamma$ and $\mathfrak{M}(\Gamma^*), \Lambda \nVdash \varphi$. This shows that $\Gamma \nvDash \varphi$. □

Problema sc.4. Show that if φ only contains variáveis proposicionais, $\vee$, and $\wedge$, then $\nvDash \varphi$. Use this to conclude that $\rightarrow$ is not definable in intuitionistic logic from $\vee$ and $\wedge$.

Problema sc.5. By using the completeness theorem prove that if $\vdash \varphi \vee \psi$ then $\vdash \varphi$ or $\vdash \psi$. (Hint: Assume $\mathfrak{M}_1 \nVdash \varphi$ and $\mathfrak{M}_2 \nVdash \psi$ and construct a new model $\mathfrak{M}$ such that $\mathfrak{M} \nVdash \varphi \vee \psi$.)

Problema sc.6. Show that if $\mathfrak{M}$ is a relational model using a linear order then $\mathfrak{M} \Vdash (\varphi \to \psi) \lor (\psi \to \varphi)$.

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sc.7 Decidability

Observe that the proof of the completeness theorem gives us for every $\Gamma \nvdash \varphi$ a model with an infinite number of worlds witnessing the fact that $\Gamma \nvDash \varphi$. The following proposition shows that to prove $\vDash \varphi$ it is enough to prove that $\mathfrak{M} \Vdash \varphi$ for all finite models (O.s., models with a finite set of worlds). int:sc:dec:sec

Teorema sc.8. *If $\nvDash \varphi$ then there is a finite model $\mathfrak{M}' \nVdash \varphi$.* *int:sc:dec:thm:decidability*

Prova. Assume $\mathfrak{M} = \langle W, R, V \rangle$ is such that $\mathfrak{M} \nVdash \varphi$ and P is the set of variáveis proposicionais occurring in φ. Define $\mathfrak{M}' = \langle W', R', V' \rangle$ by letting $W' = \{[w] : w \in W\}$ where $[w] = \{p \in P : w \in V(p)\}$, R' be the subset relation, and $V'(p) = \{[w] : p \in [w]\}$. It should be clear that W' is a finite set and that $\mathfrak{M}'$ is a modelo relacional.

It can be shown, by induction on φ, that

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}', [w] \Vdash \varphi$$

for all fórmulas φ with only variáveis proposicionais from P. This is left as an exercise for the reader. □

Problema sc.7. Finish the proof of Teorema sc.8 by showing that $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}', [w] \Vdash \varphi$ for all fórmulas φ with only propositional variables from P.

From Teorema sc.8 it follows that there is an algorithm to decide whether $\vDash \varphi$.

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Referências Bibliográficas