

-NoValue-

sc.1 Lindenbaum's Lemma

int:sc:lin: The completeness theorem for intuitionistic logic is proved by assuming $\Gamma \not\vdash \varphi$
sec and constructing a model $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \not\models \varphi$.

In classical logic the relation of **derivabilidade** can be reduced to the notion of consistency since a **fórmula** φ is **derivável** from a set of **fórmulas** iff the set together with the negation of φ is inconsistent. This is not possible in intuitionistic logic. In intuitionistic logic, if $\neg\varphi$ is inconsistent, we only get that $\vdash \neg\neg\varphi$. Since $\neg\neg\varphi \rightarrow \varphi$ does not hold intuitionistically in general, we cannot conclude that $\vdash \varphi$.

Thus, when constructing the model $\mathfrak{M}$, we will need to keep track of the non-**derivabilidade** of the **fórmula** φ and thus we will not be able to use a complete set $\Gamma^* \supseteq \Gamma$ to build the model $\mathfrak{M}$, as in every complete set Γ^* , we have $\Gamma^* \vdash \varphi \vee \neg\varphi$.

Instead of using a complete set Γ^* , we will use the notion of a prime set of formulas:

int:sc:lin: **Definição sc.1.** A set of **fórmulas** Γ is *prime* iff

defn:prime

int:sc:lin:

defn:prime1

int:sc:lin:

defn:prime2

int:sc:lin:

defn:prime3

1. Γ is consistent, O.s., $\Gamma \not\vdash \perp$;

2. if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$; and

3. if $\varphi \vee \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

int:sc:lin: **Lema sc.2 (Lindenbaum's Lemma).** If $\Gamma \not\vdash \varphi$, there is a $\Gamma^* \supseteq \Gamma$ such
lem:lindenbaum that Γ^* is prime and $\Gamma^* \not\vdash \varphi$.

Prova. Let $\psi_1 \vee \chi_1, \psi_2 \vee \chi_2, \dots$, be an enumeration of all **fórmulas** of the form $\psi \vee \chi$. We'll define an increasing sequence of sets of **fórmulas** Γ_n , where each Γ_{n+1} is defined as Γ_n together with one new **fórmula**. Γ^* will be the union of all Γ_n . The new **fórmulas** are selected so as to ensure that Γ^* is prime and still $\Gamma^* \not\vdash \varphi$. This means that at each step we should find the first disjunction $\psi_i \vee \chi_i$ such that:

int:sc:lin:

gamma-1

int:sc:lin:

gamma-2

1. $\Gamma_n \vdash \psi_i \vee \chi_i$

2. $\psi_i \notin \Gamma_n$ and $\chi_i \notin \Gamma_n$

We add to Γ_n either ψ_i if $\Gamma_n \cup \{\psi_i\} \not\vdash \varphi$, or χ_i otherwise. We'll have to show that this works. For now, let's define $i(n)$ as the least i such that (1) and (2) hold.

Define $\Gamma_0 = \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_{i(n)}\} & \text{if } \Gamma_n \cup \{\psi_{i(n)}\} \not\vdash \varphi \\ \Gamma_n \cup \{\chi_{i(n)}\} & \text{otherwise} \end{cases}$$

If $i(n)$ is undefined, O.s., whenever $\Gamma_n \vdash \psi \vee \chi$, either $\psi \in \Gamma_n$ or $\chi \in \Gamma_n$, we let $\Gamma_{n+1} = \Gamma_n$. Now let $\Gamma^* = \bigcup_{n=0}^{\infty} \Gamma_n$.

First we show that for all n , $\Gamma_n \not\vdash \varphi$. We proceed by induction on n . For $n = 0$ the claim holds by the hypothesis of the theorem, O.s., $\Gamma \not\vdash \varphi$. If $n > 0$, we have to show that if $\Gamma_n \not\vdash \varphi$ then $\Gamma_{n+1} \not\vdash \varphi$. If $i(n)$ is undefined, $\Gamma_{n+1} = \Gamma_n$ and there is nothing to prove. So suppose $i(n)$ is defined. For simplicity, let $i = i(n)$.

We'll prove the contrapositive of the claim. Suppose $\Gamma_{n+1} \vdash \varphi$. By construction, $\Gamma_{n+1} = \Gamma_n \cup \{\psi_i\}$ if $\Gamma_n \cup \{\psi_i\} \not\vdash \varphi$, or else $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. It clearly can't be the first, since then $\Gamma_{n+1} \not\vdash \varphi$. Hence, $\Gamma_n \cup \{\psi_i\} \vdash \varphi$ and $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. By definition of $i(n)$, we have that $\Gamma_n \vdash \psi_i \vee \chi_i$. We have $\Gamma_n \cup \{\psi_i\} \vdash \varphi$. We also have $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\} \vdash \varphi$. Hence, $\Gamma_n \vdash \varphi$, which is what we wanted to show.

If $\Gamma^* \vdash \varphi$, there would be some finite subset $\Gamma' \subseteq \Gamma^*$ such that $\Gamma' \vdash \varphi$. Each $\theta \in \Gamma'$ must be in Γ_i for some i . Let n be the largest of these. Since $\Gamma_i \subseteq \Gamma_n$ if $i \leq n$, $\Gamma' \subseteq \Gamma_n$. But then $\Gamma_n \vdash \varphi$, contrary to our proof above that $\Gamma_n \not\vdash \varphi$.

Lastly, we show that Γ^* is prime, O.s., satisfies conditions (1), (2), and (3) of **Definição sc.1**.

First, $\Gamma^* \not\vdash \varphi$, so Γ^* is consistent, so (1) holds.

We now show that if $\Gamma^* \vdash \psi \vee \chi$, then either $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$. This proves (3), since if $\psi \vee \chi \in \Gamma^*$ then also $\Gamma^* \vdash \psi \vee \chi$. So assume $\Gamma^* \vdash \psi \vee \chi$ but $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$. Since $\Gamma^* \vdash \psi \vee \chi$, $\Gamma_n \vdash \psi \vee \chi$ for some n . $\psi \vee \chi$ appears on the enumeration of all disjunctions, say, as $\psi_j \vee \chi_j$. $\psi_j \vee \chi_j$ satisfies the properties in the definition of $i(n)$, namely we have $\Gamma_n \vdash \psi_j \vee \chi_j$, while $\psi_j \notin \Gamma_n$ and $\chi_j \notin \Gamma_n$. At each stage, at least one fewer disjunction $\psi_i \vee \chi_i$ satisfies the conditions (since at each stage we add either ψ_i or χ_i), so at some stage m we will have $j = i(m)$. But then either $\psi \in \Gamma_{m+1}$ or $\chi \in \Gamma_{m+1}$, contrary to the assumption that $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$.

Now suppose $\Gamma^* \vdash \psi$. Then $\Gamma^* \vdash \psi \vee \psi$. But we've just proved that if $\Gamma^* \vdash \psi \vee \psi$ then $\psi \in \Gamma^*$. Hence, Γ^* satisfies (2) of **Definição sc.1**. $\square$

Problema sc.1. Show that if $\Gamma \not\vdash \perp$ then Γ is consistent in classical logic, O.s., there is a valuation making all formulas in Γ true.

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Referências Bibliográficas