-NoValue-

pty.1 Converting Derivações to Proof Terms

int:pty:pt:sec

We will describe the process of converting natural deduction derivações to pairs. We will write a proof term to the left of each formula in the derivação, resulting in expressions of the form $M : \varphi$. We'll then say that, M *witnesses* φ. Let's call such an expression a *judgment*.

First let us assign to each assumption a variable, with the following constraints:

1. Assumptions descarregada in the same step (that is, with the same number on the square bracket) must be assigned the same variable.

2. For assumptions not descarregada, assumptions of different fórmulas should be assigned different variables.

Such an assignment translates all assumptions of the form

$$\varphi \qquad \text{into} \qquad x : \varphi.$$

With assumptions all associated with variables (which are terms), we can now inductively translate the rest of the deduction tree. The modified natural deduction rules taking into account context and proof terms are given below. Given the proof terms for the premise(s), we obtain the corresponding proof term for conclusion.

$$\frac{M_1 : \varphi_1 \qquad M_2 : \varphi_2}{\langle M_1, M_2 \rangle : \varphi_1 \wedge \varphi_2} \wedge\text{Intro}$$

$$\frac{M : \varphi_1 \wedge \varphi_2}{\mathrm{p}_i(M) : \varphi_1} \wedge\text{Elim}_1 \qquad \frac{M : \varphi_1 \wedge \varphi_2}{\mathrm{p}_i(M) : \varphi_2} \wedge\text{Elim}_2$$

In $\wedge$Intro we assume we have φ_1 witnessed by term M_1 and φ_2 witnessed by term M_2. We pack up the two terms into a pair $\langle M_1, M_2 \rangle$ which witnesses $\varphi_1 \wedge \varphi_2$.

In $\wedge\text{Elim}_i$ we assume that M witnesses $\varphi_1 \wedge \varphi_2$. The term witnessing φ_i is $\mathrm{p}_i(M)$. Note that M is not necessary of the form $\langle M_1, M_2 \rangle$, so we cannot simply assign M_1 to the conclusion φ_i.

Note how this coincides with the BHK interpretation. What the BHK interpretation does not specify is how the function used as proof for $\varphi \to \psi$ is supposed to be obtained. If we think of proof terms as proofs or functions of proofs, we can be more explicit.

$$\frac{\begin{array}{c}[x : \varphi] \\ \vdots \\ N : \psi\end{array}}{\lambda x^{\varphi}.\, N : \varphi \to \psi} \to\text{Intro} \qquad \frac{P : \varphi \to \psi \qquad Q : \varphi}{PQ : \psi} \to\text{Elim}$$

The λ notation should be understood as the same as in the lambda calculus, and PQ means applying P to Q.

$$\dfrac{M_1 : \varphi_1}{\mathrm{in}_1^{\varphi_1}(M_1) : \varphi_1 \lor \varphi_2}\,\lor\mathrm{Intro}_1 \qquad \dfrac{M_2 : \varphi_2}{\mathrm{in}_2^{\varphi_2}(M_2) : \varphi_1 \lor \varphi_2}\,\lor\mathrm{Intro}_2$$

$$\dfrac{M : A_1 \lor \varphi_2 \qquad \begin{matrix}[x_1 : \varphi_1] \\ \vdots \\ N_1 : \chi\end{matrix} \qquad \begin{matrix}[x_2 : \varphi_2] \\ \vdots \\ N_2 : \chi\end{matrix}}{\mathrm{case}(M, x_1.N_1, x_2.N_2) : \chi}\,\lor\mathrm{Elim}$$

The proof term $\mathrm{in}_1^{\varphi_1}(M_1)$ is a term witnessing $\varphi_1 \lor \varphi_2$, where M_1 witnesses φ_1.

The term $\mathrm{case}(M, x_1.N_1, x_2.N_2)$ mimics the case clause in programming languages: we already have the derivação of $\varphi \lor \psi$, a derivação of χ assuming φ, and a derivação of χ assuming ψ. The *case* operator thus select the appropriate proof depending on M; either way it's a proof of χ.

$$\dfrac{N : \bot}{\mathrm{contr}_\varphi(N) : \varphi}\,\bot_I$$

$\mathrm{contr}_\varphi(N)$ is a term witnessing φ, whenever N is a term witnessing $\bot$.

Now we have a natural deduction derivação with all formulas associated with a term. At each step, the relevant typing context Γ is given by the list of assumptions remaining não descarregada at that step. Note that Γ is well defined: since we have forbidden assumptions of different não descarregada assumptions to be assigned the same variable, there won't be any disagreement about the formulas mapped to which a variable is mapped.

We now give some examples of such translations:

Consider the derivação of $\neg\neg(\varphi \lor \neg\varphi)$, O.s., $((\varphi \lor (\varphi \to \bot)) \to \bot) \to \bot$. Its translation is:

$$2\,\dfrac{\dfrac{[y : (\varphi \lor (\varphi \to \bot)) \to \bot]^2 \qquad \dfrac{1\,\dfrac{\dfrac{[y : (\varphi \lor (\varphi \to \bot)) \to \bot]^2 \qquad \dfrac{[x : \varphi]^1}{\mathrm{in}_1^{\varphi \to \bot}(x) : \varphi \lor (\varphi \to \bot)}}{y(\mathrm{in}_1^{\varphi \to \bot}(x)) : \bot}}{\lambda x^\varphi.\, y(\mathrm{in}_1^{\varphi \to \bot}(x)) : \varphi \to \bot}}{\mathrm{in}_2^\varphi(\lambda x^\varphi.\, y(\mathrm{in}_1^{\varphi \to \bot}(x))) : \varphi \lor (\varphi \to \bot)}}{y(\mathrm{in}_2^\varphi(\lambda x^\varphi.\, y\mathrm{in}_1^{\varphi \to \bot}(x))) : \bot}}{\lambda y^{(\varphi \lor (\varphi \to \bot)) \to \bot}.\, y(\mathrm{in}_2^\varphi(\lambda x^\varphi.\, y\mathrm{in}_1^{\varphi \to \bot}(x))) : ((\varphi \lor (\varphi \to \bot)) \to \bot) \to \bot}$$

The tree has no assumptions, so the context is empty; we get:

$$\vdash \lambda y^{(\varphi \lor (\varphi \to \bot)) \to \bot}.\, y(\mathrm{in}_2^\varphi(\lambda x^\varphi.\, y\mathrm{in}_1^{\varphi \to \bot}(x))) : ((\varphi \lor (\varphi \to \bot)) \to \bot) \to \bot$$

If we leave out the last →Intro, the assumptions denoted by y would be in the context and we would get:

$$y : ((\varphi \vee (\varphi \to \bot)) \to \bot) \vdash y(\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y\mathrm{in}_1^{\varphi \to \bot}(x))) : \bot$$

Another example: $\vdash \varphi \to (\varphi \to \bot) \to \bot$

$$\cfrac{\cfrac{\cfrac{[x : \varphi]^2 \qquad [y : \varphi \to \bot]^1}{yx : \bot}\;1}{\lambda y^{\varphi \to \bot}.\, yx : (\varphi \to \bot) \to \bot}\;2}{\lambda x^{\varphi}.\, \lambda y^{\varphi \to \bot}.\, yx : \varphi \to (\varphi \to \bot) \to \bot}$$

Again all assumptions are descarregada and thus the context is empty, the resulting term is

$$\vdash \lambda x^{\varphi}.\, \lambda y^{\varphi \to \bot}.\, yx : \varphi \to (\varphi \to \bot) \to \bot$$

If we leave out the last two →Intro inferences, the assumptions denoted by both x and y would be in context and we would get

$$x : \varphi, y : \varphi \to \bot \vdash yx : \bot$$

Créditos das Imagens

Referências Bibliográficas