

-NoValue-

tcp.1 Theories in which $\mathbf{Q}$ is Interpretable are Undecidable

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We can strengthen these results even more. Informally, an interpretation of a language $\mathcal{L}_1$ in another language $\mathcal{L}_2$ involves defining the universe, relation symbols, and function symbols of $\mathcal{L}_1$ with *fórmulas* in $\mathcal{L}_2$. Though we won't take the time to do this, one can make this definition precise.

Teorema tcp.1. *Suppose $\mathbf{T}$ is a theory in a language in which one can interpret the language of arithmetic, in such a way that $\mathbf{T}$ is consistent with the interpretation of $\mathbf{Q}$. Then $\mathbf{T}$ is undecidable. If $\mathbf{T}$ proves the interpretation of the axioms of $\mathbf{Q}$, then no consistent extension of $\mathbf{T}$ is decidable.*

The proof is just a small modification of the proof of the last theorem; one could use a counterexample to get a separation of $\mathbf{Q}$ and $\bar{\mathbf{Q}}$. One can take **ZFC**, Zermelo–Fraenkel set theory with the axiom of choice, to be an axiomatic foundation that is powerful enough to carry out a good deal of ordinary mathematics. In **ZFC** one can define the natural numbers, and via this interpretation, the axioms of $\mathbf{Q}$ are true. So we have

Corolário tcp.2. *There is no decidable extension of **ZFC**.*

Corolário tcp.3. *There is no complete, consistent, computably *axiomatizável* extension of **ZFC**.*

The language of **ZFC** has only a single binary relation, $\in$. (In fact, you don't even need equality.) So we have

Corolário tcp.4. *First-order logic for any language with a binary relation symbol is undecidable.*

This result extends to any language with two unary function symbols, since one can use these to simulate a binary relation symbol. The results just cited are tight: it turns out that first-order logic for a language with only *unary* relation symbols and at most one *unary* function symbol is decidable.

One more bit of trivia. We know that the set of sentences in the language $0, ', +, \times, <$ true in the standard model is undecidable. In fact, one can define $<$ in terms of the other symbols, and then one can define $+$ in terms of $\times$ and $'$. So the set of true sentences in the language $0, ', \times$ is undecidable. On the other hand, Presburger has shown that the set of sentences in the language $0, ', +$ true in the language of arithmetic is decidable. The procedure is computationally infeasible, however.

Créditos das Imagens

Referências Bibliográficas