-NoValue-

req.1 Undecidability

inc:req:und:sec We call a theory $\mathbf{T}$ *undecidable* if there is no computational procedure which, after finitely many steps and unfailingly, provides a correct answer to the question "does $\mathbf{T}$ prove φ?" for any sentence φ in the language of $\mathbf{T}$. So $\mathbf{Q}$ would be decidable iff there were a computational procedure which decides, given a sentence φ in the language of arithmetic, whether $\mathbf{Q} \vdash \varphi$ or not. We can make this more precise by asking: Is the relation $\mathrm{Prov}_{\mathbf{Q}}(y)$, which holds of y iff y is the Gödel number of a sentence provable in $\mathbf{Q}$, recursive? The answer is: no.

Teorema req.1. $\mathbf{Q}$ *is undecidable, O.s., the relation*

$$\mathrm{Prov}_{\mathbf{Q}}(y) \Leftrightarrow \mathrm{Sent}(y) \wedge \exists x\, \mathrm{Prf}_{\mathbf{Q}}(x, y)$$

is not recursive.

Prova. Suppose it were. Then we could solve the halting problem as follows: Given e and n, we know that $\varphi_e(n)\downarrow$ iff there is an s such that $T(e, n, s)$, where T is Kleene's predicate from **??**. Since T is primitive recursive it is representable in $\mathbf{Q}$ by a formula ψ_T, that is, $\mathbf{Q} \vdash \psi_T(\overline{e}, \overline{n}, \overline{s})$ iff $T(e, n, s)$. If $\mathbf{Q} \vdash \psi_T(\overline{e}, \overline{n}, \overline{s})$ then also $\mathbf{Q} \vdash \exists y\, \psi_T(\overline{e}, \overline{n}, y)$. If no such s exists, then $\mathbf{Q} \vdash \neg\psi_T(\overline{e}, \overline{n}, \overline{s})$ for every s. But $\mathbf{Q}$ is ω-consistent, O.s., if $\mathbf{Q} \vdash \neg\varphi(\overline{n})$ for every $n \in \mathbb{N}$, then $\mathbf{Q} \nvdash \exists y\, \varphi(y)$. We know this because the axioms of $\mathbf{Q}$ are true in the standard model $\mathfrak{N}$. So, $\mathbf{Q} \nvdash \exists y\, \psi_T(\overline{e}, \overline{n}, y)$. In other words, $\mathbf{Q} \vdash \exists y\, \psi_T(\overline{e}, \overline{n}, y)$ iff there is an s such that $T(e, n, s)$, O.s., iff $\varphi_e(n)\downarrow$. From e and n we can compute $^{\#}\exists y\, \psi_T(\overline{e}, \overline{n}, y)^{\#}$, let $g(e, n)$ be the primitive recursive function which does that. So

$$h(e, n) = \begin{cases} 1 & \text{if } \mathrm{Prov}_{\mathbf{Q}}(g(e, n)) \\ 0 & \text{otherwise.} \end{cases}$$

This would show that h is recursive if $\mathrm{Prov}_{\mathbf{Q}}$ is. But h is not recursive, by **??**, so $\mathrm{Prov}_{\mathbf{Q}}$ cannot be either. □

Corolário req.2. *First-order logic is undecidable.*

Prova. If first-order logic were decidable, provability in $\mathbf{Q}$ would be as well, since $\mathbf{Q} \vdash \varphi$ iff $\vdash \tau \to \varphi$, where τ is the conjunction of the axioms of $\mathbf{Q}$. □

Créditos das Imagens

Referências Bibliográficas