

-NoValue-

inp.1 Σ_1 completeness

inc:inp:slc:sec Despite the incompleteness of $\mathbf{Q}$ and its consistent, axiomatizable extensions, we have seen that $\mathbf{Q}$ does prove many basic facts about numerals. In fact, this can be extended quite considerably. To understand the scope of what can be proved in $\mathbf{Q}$, we introduce the notions of Δ_0 , Σ_1 , and Π_1 *fórmulas*. Roughly speaking, a Σ_1 *fórmula* is one of the form $\exists x \psi(x)$, where ψ is constructed using only propositional connectives and bounded quantifiers. We shall show that if φ is a Σ_1 *sentença* which is true in $\mathfrak{N}$, then $\mathbf{Q} \vdash \varphi$ (**Teorema inp.7**).

inc:inp:slc:defn:bd-quant **Definição inp.1.** A *bounded existential fórmula* is one of the form $\exists x (x < t \wedge \varphi(x))$ where t is any term, which we conventionally write as $(\exists x < t) \varphi(x)$. A *bounded universal fórmula* is one of the form $\forall x (x < t \rightarrow \varphi(x))$ where t is any term, which we conventionally write as $(\forall x < t) \varphi(x)$.

inc:inp:slc:defn:delta0-sigma1-pi1-frn **Definição inp.2.** A *fórmula* ψ is Δ_0 if it is built up from atomic *fórmulas* using only propositional connectives and bounded quantification. A *fórmula* φ is Σ_1 if $\varphi \equiv \exists x \psi(x)$ where ψ is Δ_0 . A *fórmula* φ is Π_1 if $\varphi \equiv \forall x \psi(x)$ where ψ is Δ_0 .

inc:inp:slc:lem:q-proves-clterm-id **Lema inp.3.** Suppose t is a closed term such that $\text{Val}^{\mathfrak{N}}(t) = n$. Then $\mathbf{Q} \vdash t = \bar{n}$.

Prova. We prove this by induction on the complexity of t . For the base case, $\text{Val}^{\mathfrak{N}}(0) = 0$, and $\mathbf{Q} \vdash 0 = \bar{0}$ since $\bar{0} \equiv 0$. For the inductive case, let t_1 and t_2 be terms such that $\text{Val}^{\mathfrak{N}}(t_1) = n_1$, $\text{Val}^{\mathfrak{N}}(t_2) = n_2$, $\mathbf{Q} \vdash t_1 = \bar{n}_1$, and $\mathbf{Q} \vdash t_2 = \bar{n}_2$.

Then $\text{Val}^{\mathfrak{N}}((t'_1)) = n_1 + 1$, and we have that $\mathbf{Q} \vdash t'_1 = \bar{n}_1'$ by the first-order rules for identity applied to the induction hypothesis and the *fórmula* $\bar{n}_1' = \overline{n_1'}$, so we have $\mathbf{Q} \vdash t'_1 = \bar{n}_1 + \bar{1}$ by the definition of numerals.

For sums we have

$$\text{Val}^{\mathfrak{N}}((t_1 + t_2)) = \text{Val}^{\mathfrak{N}}(t_1) + \text{Val}^{\mathfrak{N}}(t_2) = n_1 + n_2.$$

By the induction hypothesis and the rules for identity, $\mathbf{Q} \vdash t_1 + t_2 = \bar{n}_1 + t_2$, and then $\mathbf{Q} \vdash t_1 + t_2 = \overline{\bar{n}_1 + n_2}$ by a second application of the rules for identity. By ??, $\mathbf{Q} \vdash \bar{n}_1 + \bar{n}_2 = \overline{n_1 + n_2}$, so $\mathbf{Q} \vdash t_1 + t_2 = \overline{n_1 + n_2}$.

Similar reasoning also works for $\times$, using ??. Since this exhausts the closed terms of arithmetic, we have that $\mathbf{Q} \vdash t = \bar{n}$ for all closed terms t such that $\text{Val}^{\mathfrak{N}}(t) = n$. $\square$

Problema inp.1. Prove in detail the part of **Lema inp.3** involving $\times$.

inc:inp:slc:lem:atomic-completeness **Lema inp.4.** Suppose t_1 and t_2 are closed terms. Then

1. If $\text{Val}^{\mathfrak{N}}(t_1) = \text{Val}^{\mathfrak{N}}(t_2)$, then $\mathbf{Q} \vdash t_1 = t_2$.

2. If $\text{Val}^{\mathfrak{N}}(t_1) \neq \text{Val}^{\mathfrak{N}}(t_2)$, then $\mathbf{Q} \vdash t_1 \neq t_2$.
3. If $\text{Val}^{\mathfrak{N}}(t_1) < \text{Val}^{\mathfrak{N}}(t_2)$, then $\mathbf{Q} \vdash t_1 < t_2$.
4. If $\text{Val}^{\mathfrak{N}}(t_2) \leq \text{Val}^{\mathfrak{N}}(t_1)$, then $\mathbf{Q} \vdash \neg(t_1 < t_2)$.

Prova. Given terms t_1 and t_2 , we fix $n = \text{Val}^{\mathfrak{N}}(t_1)$ and $m = \text{Val}^{\mathfrak{N}}(t_2)$.

Suppose $\varphi \equiv t_1 = t_2$. By [Lema inp.3](#), $\mathbf{Q} \vdash t_1 = \bar{n}$ and $\mathbf{Q} \vdash t_2 = \bar{m}$. If $n = m$, then $\mathbf{Q} \vdash \bar{n} = \bar{m}$ and hence $\mathbf{Q} \vdash t_1 = t_2$ by the transitivity of identity. If $n \neq m$ then $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$, and by the transitivity of identity again, $\mathbf{Q} \vdash t_1 \neq t_2$.

Now let $\varphi \equiv t_1 < t_2$. For both cases, we rely on axiom Q_8 , which states that $x < y \leftrightarrow \exists z z' + x = y$ for all x, y .

Suppose $\mathfrak{N} \models t_1 < t_2$. Then there exists some $k \in \mathbb{N}$ such that $n + k + 1 = m$. By [Lema inp.3](#), $\mathbf{Q} \vdash t_1 = \bar{n}$ and $\mathbf{Q} \vdash t_2 = \bar{m}$, and by the first part of this lemma, $\mathbf{Q} \vdash \bar{n} + \bar{k}' = \bar{m}$. By the transitivity of identity it follows that $\mathbf{Q} \vdash \bar{k}' + t_1 = t_2$, so $\mathbf{Q} \vdash \exists z z' + t_1 = t_2$. By the right-to-left direction of Q_8 , $\mathbf{Q} \vdash t_1 < t_2$.

Suppose instead that $\mathfrak{N} \not\models t_1 < t_2$, O.s., $m \leq n$. We work in $\mathbf{Q}$ and assume that $t_1 < t_2$. By the left-to-right direction of Q_8 , there is some z such that $z' + t_1 = t_2$. Since $\mathbf{Q} \vdash t_1 = \bar{n}$ and $\mathbf{Q} \vdash t_2 = \bar{m}$, $z' + \bar{n} = \bar{m}$. By an external induction on m using Q_5 , $z' + \bar{n} - \bar{m} = 0$. If $m = n$ then $z' \neq 0$, giving a contradiction via Q_3 . If $m < n$ then $(z' + \bar{n} - \bar{m} - 1)' = 0$ by Q_5 again, giving a contradiction via Q_3 . So $\mathbf{Q} \vdash \neg(t_1 < t_2)$. $\square$

Lema inp.5. Suppose φ is a *fórmula*, t a closed term, and $k = \text{Val}^{\mathfrak{N}}(t)$. Then

*inc:inp:s1c:
lem:bounded-quant-equiv*

1. $\mathbf{Q} \vdash (\forall x < t) \varphi(x)$ iff $\mathbf{Q} \vdash \varphi(\bar{0}) \wedge \dots \wedge \varphi(\overline{k-1})$.
2. $\mathbf{Q} \vdash (\exists x < t) \varphi(x)$ iff $\mathbf{Q} \vdash \varphi(\bar{0}) \vee \dots \vee \varphi(\overline{k-1})$.

Prova. We prove the case for the bounded universal quantifier. If $\text{Val}^{\mathfrak{N}}(t) = 0$ then the left-hand side of the equivalence is provable in $\mathbf{Q}$, because there is no $x < \bar{0}$ by Q_1 . Similarly, we can take an empty disjunction to be simply $\top$, which is also provable in $\mathbf{Q}$. We therefore suppose that $\text{Val}^{\mathfrak{N}}(t) = k + 1$ for some natural number k . By [Lema inp.3](#) we can assume that we are working with a *fórmula* of the form $(\forall x < \overline{k+1}) \varphi(x)$.

Suppose that $\mathbf{Q} \vdash (\forall x < \overline{k+1}) \varphi(x)$, and let $n \leq k$. Since $\mathbf{Q} \vdash \bar{n} < \overline{k+1}$ by [Lema inp.4](#), it follows by logic that $\mathbf{Q} \vdash \varphi(\bar{n})$. Applying this fact $k+1$ times for each $n \leq k$, we get that $\mathbf{Q} \vdash \varphi(\bar{0}) \wedge \dots \wedge \varphi(\bar{k})$ as desired.

For the other direction, suppose that $\mathbf{Q} \vdash \varphi(\bar{0}) \wedge \dots \wedge \varphi(\bar{k})$. Working in $\mathbf{Q}$, suppose that $x < \overline{k+1}$. By Q_1 we have that $x = \bar{0} \vee \dots \vee x = \bar{k}$, so by logic it follows that $\varphi(x)$, and hence the universal claim $(\forall x < \overline{k+1}) \varphi(x)$ follows.

The proof of the equivalence for bounded existentially quantified *fórmulas* is similar. $\square$

Problema inp.2. Give a detailed proof of the existential case in [Lema inp.5](#).

Lema inp.6. If φ is a Δ_0 *sentença* which is true in $\mathfrak{N}$, then $\mathbf{Q} \vdash \varphi$.

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lem:delta0-completeness*

Prova. We prove this by induction on **fórmula** complexity. The base case is given by **Lema inp.4**, so we move to the induction step. For simplicity we split the case of negation into subcases depending on the structure of the **fórmula** to which the negation is applied.

1. Suppose $(\varphi \wedge \psi)$ is true in $\mathfrak{N}$, so φ and ψ are true in $\mathfrak{N}$. By the induction hypothesis, $\mathbf{Q} \vdash \varphi$ and $\mathbf{Q} \vdash \psi$, so $\mathbf{Q} \vdash (\varphi \wedge \psi)$ by logic.
2. Suppose $\neg(\varphi \wedge \psi)$ is true in $\mathfrak{N}$, so either $\neg\varphi$ or $\neg\psi$ is true in $\mathfrak{N}$. Without loss of generality, suppose the former. By the induction hypothesis $\mathbf{Q} \vdash \neg\varphi$, and hence $\mathbf{Q} \vdash \neg(\varphi \wedge \psi)$ by logic.
3. Suppose $(\varphi \vee \psi)$ is true in $\mathfrak{N}$, so either φ is true in $\mathfrak{N}$ or ψ is true in $\mathfrak{N}$. Without loss of generality, suppose the former holds. By the induction hypothesis $\mathbf{Q} \vdash \varphi$, and hence $\mathbf{Q} \vdash (\varphi \vee \psi)$ by logic.
4. Suppose $\neg(\varphi \vee \psi)$ is true in $\mathfrak{N}$, so $\neg\varphi$ and $\neg\psi$ are true in $\mathfrak{N}$. Then $\mathbf{Q} \vdash \neg\varphi$ and $\mathbf{Q} \vdash \neg\psi$ by the induction hypothesis. Consequently, $\mathbf{Q} \vdash \neg(\varphi \vee \psi)$ by logic.
5. Suppose that $(\forall x < t) \varphi(x)$ is true in $\mathfrak{N}$, where t is a closed term and $k = \text{Val}^{\mathfrak{N}}(t)$. By the induction hypothesis and logic, if $\varphi(\bar{n})$ is true in $\mathfrak{N}$ for all $n < \text{Val}^{\mathfrak{N}}(t)$ then $\mathbf{Q} \vdash \varphi(\bar{0}) \wedge \dots \wedge \varphi(\bar{k-1})$. By **Lema inp.5** it follows that $\mathbf{Q} \vdash (\forall x < t) \varphi(x)$.
6. The case for the bounded existential quantifier, where we have a **sentença** of the form $(\exists x < t) \varphi(x)$, is similar to that for the bounded universal quantifier.
7. Suppose that $\neg(\forall x < t) \varphi(x)$ is true in $\mathfrak{N}$, where t is a closed term. This **sentença** is equivalent to the **sentença** $(\exists x < t) \neg\varphi(x)$, with the equivalence derivable in $\mathbf{Q}$, so we may apply the reasoning for bounded existential quantifiers.
8. Similarly, suppose that $\neg(\exists x < t) \varphi(x)$ is true in $\mathfrak{N}$, where t is a closed term. This **sentença** is equivalent in $\mathbf{Q}$ to $(\forall x < t) \neg\varphi(x)$, and so we may apply the reasoning for bounded universal quantifiers.
9. Finally, suppose $\neg\varphi$ is true in $\mathfrak{N}$. The only cases remaining are when φ is atomic and when $\neg\varphi \equiv \neg\neg\psi$ for some Δ_0 **sentença** ψ . If φ is atomic then by **Lema inp.4**, $\mathbf{Q} \vdash \neg\varphi$. If $\neg\varphi \equiv \neg\neg\psi$, then by logic it is provably equivalent in $\mathbf{Q}$ to ψ , which is true in $\mathfrak{N}$ since $\neg\varphi$ is true in $\mathfrak{N}$. By the induction hypothesis we therefore have that $\mathbf{Q} \vdash \neg\varphi$. $\square$

Problema inp.3. Give a detailed proof of the existential case in **Lema inp.6**.

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thm:sigma1-completeness*

Teorema inp.7. If φ is a Σ_1 **sentença** which is true in $\mathfrak{N}$, then $\mathbf{Q} \vdash \varphi$.

Prova. If $\exists x\varphi(x)$ is a Σ_1 **sentença** which is true in $\mathfrak{N}$, then there exists a natural number n and a variable assignment s such that $s(x) = n$ and $\mathfrak{N}, s \models \varphi(x)$. By standard facts about the satisfaction relation it follows that $\mathfrak{N} \models \varphi(\bar{n})$. But $\varphi(\bar{n})$ is a Δ_0 **fórmula**, so by **Lema inp.6** we have that $\mathbf{Q} \vdash \varphi(\bar{n})$, and hence by logic we also have that $\mathbf{Q} \vdash \exists x\varphi(x)$. $\square$

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Referências Bibliográficas