-NoValue-

req.1 Functions Representable in Q are Computable

inc:req:rpc:sec We'll prove that every function that is representable in **Q** is computable. We first have to establish a lemma about functions representable in **Q**.

inc:req:rpc:lem:rep-q **Lema req.1.** *If $f(x_0, \dots, x_k)$ is representable in* **Q***, there is a fórmula $\varphi(x_0, \dots, x_k, y)$ such that*

$$\mathbf{Q} \vdash \varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{m}) \quad \textit{iff} \quad m = f(n_0, \dots, n_k).$$

Prova. The "if" part is **????**. The "only if" part is seen as follows: Suppose $\mathbf{Q} \vdash \varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{m})$ but $m \neq f(n_0, \dots, n_k)$. Let $l = f(n_0, \dots, n_k)$. By **????**, $\mathbf{Q} \vdash \varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{l})$. By **????**, $\forall y\, (\varphi_f(\overline{n_0}, \dots, \overline{n_k}, y) \rightarrow \overline{l} = y)$. Using logic and the assumption that $\mathbf{Q} \vdash \varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{m})$, we get that $\mathbf{Q} \vdash \overline{l} = \overline{m}$. On the other hand, by **??**, $\mathbf{Q} \vdash \overline{l} \neq \overline{m}$. So **Q** is inconsistent. But that is impossible, since **Q** is satisfied by the standard model (see **??**), $\mathfrak{N} \models \mathbf{Q}$, and satisfiable theories are always consistent by the Soundness Theorem (**??????????????**). □

Lema req.2. *Every function that is representable in* **Q** *is computable.*

Prova. Let's first give the intuitive idea for why this is true. To compute f, we do the following. List all the possible derivações δ in the language of arithmetic. This is possible to do mechanically. For each one, check if it is a derivação of a fórmula of the form $\varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{m})$ (the fórmula representing f in **Q** from Lema req.1). If it is, $m = f(n_0, \dots, n_k)$ by Lema req.1, and we've found the value of f. The search terminates because $\mathbf{Q} \vdash \varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{f(n_0, \dots, n_k)})$, so eventually we find a δ of the right sort.

This is not quite precise because our procedure operates on derivações and fórmulas instead of just on numbers, and we haven't explained exactly why "listing all possible derivações" is mechanically possible. But as we've seen, it is possible to code terms, fórmulas, and derivações by Gödel numbers. We've also introduced a precise model of computation, the general recursive functions. And we've seen that the relation $\mathrm{Prf}_{\mathbf{Q}}(d, y)$, which holds iff d is the Gödel number of a derivação of the fórmula with Gödel number y from the axioms of **Q**, is (primitive) recursive. Other primitive recursive functions we'll need are num (**??**) and Subst (**??**). From these, it is possible to define f by minimization; thus, f is recursive.

First, define

$$\begin{aligned}A(n_0, \dots, n_k, m) = \\ &\mathrm{Subst}(\mathrm{Subst}(\dots \mathrm{Subst}({}^{\#}\varphi_f{}^{\#}, \mathrm{num}(n_0), {}^{\#}x_0{}^{\#}), \\ &\qquad \dots), \mathrm{num}(n_k), {}^{\#}x_k{}^{\#}), \mathrm{num}(m), {}^{\#}y^{\#})\end{aligned}$$

This looks complicated, but it's just the function $A(n_0, \dots, n_k, m) = {}^{\#}\varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{m})^{\#}$.

Now, consider the relation $R(n_0, \dots, n_k, s)$ which holds if $(s)_0$ is the Gödel number of a derivação from $\mathbf{Q}$ of $\varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{(s)_1})$:

$$R(n_0, \dots, n_k, s) \quad \text{iff} \quad \mathrm{Prf}_{\mathbf{Q}}((s)_0, A(n_0, \dots, n_k, (s)_1))$$

If we can find an s such that $R(n_0, \dots, n_k, s)$ holds, we have found a pair of numbers—$(s)_0$ and $(s)_1$—such that $(s)_0$ is the Gödel number of a derivação of $A_f(\overline{n_0}, \dots, \overline{n_k}, (s)_1)$. So looking for s is like looking for the pair d and m in the informal proof. And a computable function that "looks for" such an s can be defined by regular minimization. Note that R is regular: for every n_0, …, n_k, there is a derivação δ of $\mathbf{Q} \vdash \varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{f(n_0, \dots, n_k)})$, so $R(n_0, \dots, n_k, s)$ holds for $s = \langle {}^{\#}\delta^{\#}, f(n_0, \dots, n_k) \rangle$. So, we can write f as

$$f(n_0, \dots, n_k) = (\mu s \; R(n_0, \dots, n_k, s))_1. \qquad \square$$

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Referências Bibliográficas