-NoValue-

int.1 Undecidability and Incompleteness

inc:int:dec:sec Gödel's proof of the incompleteness theorems require arithmetization of syntax. But even without that we can obtain some nice results just on the assumption that a theory representa all decidível relations. The proof is a diagonal argument similar to the proof of the undecidability of the halting problem.

Teorema int.1. *If Γ is a consistent theory that representa every decidível relation, then Γ is not decidível.*

Prova. Suppose Γ were decidível. We show that if Γ representa every decidível relation, it must be inconsistent.

Decidível properties (one-place relations) are represented by fórmulas with one free variable. Let $\varphi_0(x)$, $\varphi_1(x)$, …, be a computable enumeration of all such fórmulas. Now consider the following set $D \subseteq \mathbb{N}$:

$$D = \{n : \Gamma \vdash \neg\varphi_n(\overline{n})\}$$

The set D is decidível, since we can test if $n \in D$ by first computing $\varphi_n(x)$, and from this $\neg\varphi_n(\overline{n})$. Obviously, substituting the term $\overline{n}$ for every free occurrence of x in $\varphi_n(x)$ and prefixing $\varphi(\overline{n})$ by $\neg$ is a mechanical matter. By assumption, Γ is decidível, so we can test if $\neg\varphi(\overline{n}) \in \Gamma$. If it is, $n \in D$, and if it isn't, $n \notin D$. So D is likewise decidível.

Since Γ representa all decidível properties, it representa D. And the fórmulas which representam D in Γ are all among $\varphi_0(x)$, $\varphi_1(x)$, …. So let d be a number such that $\varphi_d(x)$ representa D in Γ. If $d \notin D$, then, since $\varphi_d(x)$ representa D, $\Gamma \vdash \neg\varphi_d(\overline{d})$. But that means that d meets the defining condition of D, and so $d \in D$. This contradicts $d \notin D$. So by indirect proof, $d \in D$.

Since $d \in D$, by the definition of D, $\Gamma \vdash \neg\varphi_d(\overline{d})$. On the other hand, since $\varphi_d(x)$ representa D in Γ, $\Gamma \vdash \varphi_d(\overline{d})$. Hence, Γ is inconsistent. □

The preceding theorem shows that no consistent theory that representa all decidível relations can be decidível. We will show that **Q** does representam all decidível relations; this means that all theories that include **Q**, such as **PA** and **TA**, also do, and hence also are not decidível. (Since all these theories are true in the standard model, they are all consistent.) explanation

We can also use this result to obtain a weak version of the first incompleteness theorem. Any theory that is axiomatizável and completo is decidível. Consistent theories that are axiomatizável and representam all decidível properties then cannot be completo.

Teorema int.2. *If Γ is axiomatizável and completo it is decidível.*

Prova. Any inconsistent theory is decidível, since inconsistent theories contain all sentenças, so the answer to the question "is $\varphi \in \Gamma$" is always "yes," O.s., can be decided.

So suppose Γ is consistent, and furthermore is axiomatizável, and completo. Since Γ is axiomatizável, it is computavelmente enumerável. For we can enumerate all the correct derivações from the axioms of Γ by a computable function. From a correct derivação we can compute the sentença it deriva, and so together there is a computable function that enumerates all theorems of Γ. A sentença is a theorem of Γ iff $\neg\varphi$ is not a theorem, since Γ is consistent and completo. We can therefore decide if $\varphi \in \Gamma$ as follows. Enumerate all theorems of Γ. When φ appears on this list, we know that $\Gamma \vdash \varphi$. When $\neg\varphi$ appears on this list, we know that $\Gamma \nvdash \varphi$. Since Γ is completo, one of these cases eventually obtains, so the procedure eventually produces an answer. □

Corolário int.3. *If Γ is consistent, axiomatizável, and representa every decidível property, it is not completo.* *inc:int:dec: cor:incompleteness*

Prova. If Γ were completo, it would be decidível by the previous theorem (since it is axiomatizável and consistent). But since Γ representa every decidível property, it is not decidível, by the first theorem. □

Problema int.1. Show that $\mathbf{TA} = \{\varphi : \mathfrak{N} \vDash \varphi\}$ is not axiomatizável. You may assume that **TA** represents all decidable properties.

Once we have established that, p.ex., **Q**, representa all decidível properties, the corollary tells us that **Q** must be incomplete. However, its proof does not provide an example of an independent sentença; it merely shows that such a sentença must exist. For this, we have to arithmetize syntax and follow Gödel's original proof idea. And of course, we still have to show the first claim, namely that **Q** does, in fact, representam all decidível properties.

It should be noted that not every *interesting* theory is incomplete or undecidable. There are many theories that are sufficiently strong to describe interesting mathematical facts that do not satisify the conditions of Gödel's result. For instance, $\mathbf{Pres} = \{\varphi \in \mathcal{L}_{A^+} : \mathfrak{N} \vDash \varphi\}$, the set of sentenças of the language of arithmetic without $\times$ true in the standard model, is both complete and decidable. This theory is called Presburger arithmetic, and proves all the truths about natural numbers that can be formulated just with $\mathsf{0}$, $\prime$, and $+$.

Créditos das Imagens

Referências Bibliográficas