-NoValue-

inp.1 Löb's Theorem

inc:inp:lob:sec The Gödel sentence for a theory $\mathbf{T}$ is a fixed point of $\neg\mathsf{Prov}_{\mathbf{T}}(y)$, O.s., a sentença γ such that

$$\mathbf{T} \vdash \neg\mathsf{Prov}_{\mathbf{T}}(\ulcorner\gamma\urcorner) \leftrightarrow \gamma.$$

It is not derivável, because if $\mathbf{T} \vdash \gamma$, (a) by derivabilidade condition (1), $\mathbf{T} \vdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\gamma\urcorner)$, and (b) $\mathbf{T} \vdash \gamma$ together with $\mathbf{T} \vdash \neg\mathsf{Prov}_{\mathbf{T}}(\ulcorner\gamma\urcorner) \leftrightarrow \gamma$ gives $\mathbf{T} \vdash \neg\mathsf{Prov}_{\mathbf{T}}(\ulcorner\gamma\urcorner)$, and so $\mathbf{T}$ would be inconsistent. Now it is natural to ask about the status of a fixed point of $\mathsf{Prov}_{\mathbf{T}}(y)$, O.s., a sentença δ such that

$$\mathbf{T} \vdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\delta\urcorner) \leftrightarrow \delta.$$

If it were derivável, $\mathbf{T} \vdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\delta\urcorner)$ by condition (1), but the same conclusion follows if we apply modus ponens to the equivalence above. Hence, we don't get that $\mathbf{T}$ is inconsistent, at least not by the same argument as in the case of the Gödel sentence. This of course does not show that $\mathbf{T}$ *does* deriva δ.

We can make headway on this question if we generalize it a bit. The left-to-right direction of the fixed point equivalence, $\mathsf{Prov}_{\mathbf{T}}(\ulcorner\delta\urcorner) \rightarrow \delta$, is an instance of a general schema called a *reflection principle*: $\mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner) \rightarrow \varphi$. It is called that because it expresses, in a sense, that $\mathbf{T}$ can "reflect" about what it can deriva; basically it says, "If $\mathbf{T}$ can deriva φ, then φ is true," for any φ. This is true for sound theories only, of course, and this suggests that theories will in general not deriva every instance of it. So which instances can a theory (strong enough, and satisfying the derivabilidade conditions) deriva? Certainly all those where φ itself is derivável. And that's it, as the next result shows.

inc:inp:lob:thm:lob **Teorema inp.1.** *Let $\mathbf{T}$ be an axiomatizável theory extending $\mathbf{Q}$, and suppose $\mathsf{Prov}_{\mathbf{T}}(y)$ is a formula satisfying conditions P1–P3 from* **??**. *If $\mathbf{T}$ deriva $\mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner) \rightarrow \varphi$, then in fact $\mathbf{T}$ deriva φ.*

Put differently, if $\mathbf{T} \nvdash \varphi$, then $\mathbf{T} \nvdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner) \rightarrow \varphi$. This result is known as Löb's theorem.

The heuristic for the proof of Löb's theorem is a clever proof that Santa Claus exists. (If you don't like that conclusion, you are free to substitute any other conclusion you would like.) Here it is: explanation

1. Let X be the sentence, "If X is true, then Santa Claus exists."
2. Suppose X is true.
3. Then what it says holds; O.s., we have: if X is true, then Santa Claus exists.
4. Since we are assuming X is true, we can conclude that Santa Claus exists, by modus ponens from (2) and (3).

5. We have succeeded in deriving (4), "Santa Claus exists," from the assumption (2), "X is true." By conditional proof, we have shown: "If X is true, then Santa Claus exists."

6. But this is just the sentence X. So we have shown that X is true.

7. But then, by the argument (2)–(4) above, Santa Claus exists.

A formalization of this idea, replacing "is true" with "is derivável," and "Santa Claus exists" with φ, yields the proof of Löb's theorem. The trick is to apply the fixed-point lemma to the fórmula $\mathsf{Prov}_{\mathbf{T}}(y) \to \varphi$. The fixed point of that corresponds to the sentence X in the preceding sketch.

Prova de Teorema inp.1. Suppose φ is a sentença such that $\mathbf{T}$ deriva $\mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner) \to \varphi$. Let $\psi(y)$ be the fórmula $\mathsf{Prov}_{\mathbf{T}}(y) \to \varphi$, and use the fixed-point lemma to find a sentença θ such that $\mathbf{T}$ deriva $\theta \leftrightarrow \psi(\ulcorner\theta\urcorner)$. Then each of the following is derivável in $\mathbf{T}$:

inc:inp:lob: L-1
$$\theta \leftrightarrow (\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to \varphi) \tag{1}$$
θ is a fixed point of $\psi(y)$

inc:inp:lob: L-2
$$\theta \to (\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to \varphi) \tag{2}$$
from Eq. (1)

inc:inp:lob: L-3
$$\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta \to (\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to \varphi)\urcorner) \tag{3}$$
from Eq. (2) by condition P1

inc:inp:lob: L-4
$$\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to \mathsf{Prov}_{\mathbf{T}}(\ulcorner\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to \varphi\urcorner) \tag{4}$$
from Eq. (3) using condition P2

inc:inp:lob: L-5
$$\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to (\mathsf{Prov}_{\mathbf{T}}(\ulcorner\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner)\urcorner) \to \mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner)) \tag{5}$$
from Eq. (4) using P2 again

inc:inp:lob: L-6
$$\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to \mathsf{Prov}_{\mathbf{T}}(\ulcorner\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner)\urcorner) \tag{6}$$
by derivabilidade condition P3

inc:inp:lob: L-7
$$\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to \mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner) \tag{7}$$
from Eq. (5) and Eq. (6)

inc:inp:lob: L-8
$$\mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner) \to \varphi \tag{8}$$
by assumption of the theorem

inc:inp:lob: L-9
$$\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to \varphi \tag{9}$$
from Eq. (7) and Eq. (8)

inc:inp:lob: L-10
$$(\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \to \varphi) \to \theta \tag{10}$$
from Eq. (1)

inc:inp:lob: L-11
$$\theta \tag{11}$$
from Eq. (9) and Eq. (10)

inc:inp:lob: L-12
$$\mathsf{Prov}_{\mathbf{T}}(\ulcorner\theta\urcorner) \tag{12}$$
from Eq. (11) by condition P1

φ from Eq. (8) and Eq. (12) □

With Löb's theorem in hand, there is a short proof of the second incompleteness theorem (for theories having a derivabilidade predicate satisfying conditions P1–P3): if $\mathbf{T} \vdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\bot\urcorner) \to \bot$, then $\mathbf{T} \vdash \bot$. If $\mathbf{T}$ is consistent, $\mathbf{T} \nvdash \bot$. So, $\mathbf{T} \nvdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\bot\urcorner) \to \bot$, O.s., $\mathbf{T} \nvdash \mathsf{Con}_{\mathbf{T}}$. We can also apply it to show that δ, the fixed point of $\mathsf{Prov}_{\mathbf{T}}(x)$, is derivável. For since

$$\mathbf{T} \vdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\delta\urcorner) \leftrightarrow \delta$$

in particular

$$\mathbf{T} \vdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\delta\urcorner) \to \delta$$

and so by Löb's theorem, $\mathbf{T} \vdash \delta$.

Problema inp.1. Let $\mathbf{T}$ be a computably axiomatized theory, and let $\mathsf{Prov}_{\mathbf{T}}$ be a derivabilidade predicate for $\mathbf{T}$. Consider the following four statements:

1. If $T \vdash \varphi$, then $T \vdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner)$.
2. $T \vdash \varphi \rightarrow \mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner)$.
3. If $T \vdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner)$, then $T \vdash \varphi$.
4. $T \vdash \mathsf{Prov}_{\mathbf{T}}(\ulcorner\varphi\urcorner) \rightarrow \varphi$

Under what conditions are each of these statements true?

Créditos das Imagens

Referências Bibliográficas