

-NoValue-

inp.1 The Fixed-Point Lemma

inc:inp:fix:
sec The fixed-point lemma says that for any **fórmula** $\psi(x)$, there is a **sentença** φ explanation such that $\mathbf{T} \vdash \varphi \leftrightarrow \psi(\ulcorner \varphi \urcorner)$, provided $\mathbf{T}$ extends $\mathbf{Q}$. In the case of the liar sentence, we'd want φ to be equivalent (provably in $\mathbf{T}$) to “ $\ulcorner \varphi \urcorner$ is false,” O.s., the statement that $\# \varphi^\#$ is the Gödel number of a false **sentença**. To understand the idea of the proof, it will be useful to compare it with Quine's informal gloss of φ as, “yields a falsehood when preceded by its own quotation” yields a falsehood when preceded by its own quotation.” The operation of taking an expression, and then forming a sentence by preceding this expression by its own quotation may be called *diagonalizing* the expression, and the result its diagonalization. So, the diagonalization of ‘yields a falsehood when preceded by its own quotation’ is “yields a falsehood when preceded by its own quotation” yields a falsehood when preceded by its own quotation.” Now note that Quine's liar sentence is not the diagonalization of ‘yields a falsehood’ but of ‘yields a falsehood when preceded by its own quotation.’ So the property being diagonalized to yield the liar sentence itself involves diagonalization!

In the language of arithmetic, we form quotations of a **fórmula** with one free variable by computing its Gödel numbers and then substituting the standard numeral for that Gödel number into the free variable. The diagonalization of $\alpha(x)$ is $\alpha(\bar{n})$, where $n = \# \alpha(x)^\#$. (From now on, let's abbreviate $\# \alpha(x)^\#$ as $\ulcorner \alpha(x) \urcorner$.) So if $\psi(x)$ is “is a falsehood,” then “yields a falsehood if preceded by its own quotation,” would be “yields a falsehood when applied to the Gödel number of its diagonalization.” If we had a symbol *diag* for the function $\text{diag}(n)$ which computes the Gödel number of the diagonalization of the **fórmula** with Gödel number n , we could write $\alpha(x)$ as $\psi(\text{diag}(x))$. And Quine's version of the liar sentence would then be the diagonalization of it, O.s., $\alpha(\ulcorner \alpha(x) \urcorner)$ or $\psi(\text{diag}(\ulcorner \psi(\text{diag}(x)) \urcorner))$. Of course, $\psi(x)$ could now be any other property, and the same construction would work. For the incompleteness theorem, we'll take $\psi(x)$ to be “ x is not **derivável** in $\mathbf{T}$.” Then $\alpha(x)$ would be “yields a **sentença** not **derivável** in $\mathbf{T}$ when applied to the Gödel number of its diagonalization.”

To formalize this in $\mathbf{T}$, we have to find a way to formalize *diag*. The function $\text{diag}(n)$ is computable, in fact, it is primitive recursive: if n is the Gödel number of a formula $\alpha(x)$, $\text{diag}(n)$ returns the Gödel number of $\alpha(\ulcorner \alpha(x) \urcorner)$. (Recall, $\ulcorner \alpha(x) \urcorner$ is the standard numeral of the Gödel number of $\alpha(x)$, O.s., $\# \alpha(x)^\#$). If *diag* were a function symbol in $\mathbf{T}$ representing the function *diag*, we could take φ to be the formula $\psi(\text{diag}(\ulcorner \psi(\text{diag}(x)) \urcorner))$. Notice that

$$\begin{aligned} \text{diag}(\# \psi(\text{diag}(x))^\#) &= \# \psi(\text{diag}(\ulcorner \psi(\text{diag}(x)) \urcorner))^\# \\ &= \# \varphi^\#. \end{aligned}$$

Assuming $\mathbf{T}$ can **deriva**

$$\text{diag}(\ulcorner \psi(\text{diag}(x)) \urcorner) = \ulcorner \varphi \urcorner,$$

it can **deriva** $\psi(\text{diag}(\ulcorner \psi(\text{diag}(x)) \urcorner)) \leftrightarrow \psi(\ulcorner \varphi \urcorner)$. But the left hand side is, by definition, φ .

Of course, diag will in general not be a function symbol of $\mathbf{T}$, and certainly is not one of $\mathbf{Q}$. But, since diag is computable, it is *representable* in $\mathbf{Q}$ by some formula $\theta_{\text{diag}}(x, y)$. So instead of writing $\psi(\text{diag}(x))$ we can write $\exists y (\theta_{\text{diag}}(x, y) \wedge \psi(y))$. Otherwise, the proof sketched above goes through, and in fact, it goes through already in $\mathbf{Q}$.

Lema inp.1. *Let $\psi(x)$ be any formula with one free variable x . Then there is a sentence φ such that $\mathbf{Q} \vdash \varphi \leftrightarrow \psi(\ulcorner \varphi \urcorner)$.*

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Prova. Given $\psi(x)$, let $\alpha(x)$ be the formula $\exists y (\theta_{\text{diag}}(x, y) \wedge \psi(y))$ and let φ be its diagonalization, O.s., the formula $\alpha(\ulcorner \alpha(x) \urcorner)$.

Since θ_{diag} represents diag , and $\text{diag}(\ulcorner \alpha(x) \urcorner) = \ulcorner \varphi \urcorner$, $\mathbf{Q}$ can **deriva**

$$\theta_{\text{diag}}(\ulcorner \alpha(x) \urcorner, \ulcorner \varphi \urcorner) \quad (1)$$

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repdiag1*

$$\forall y (\theta_{\text{diag}}(\ulcorner \alpha(x) \urcorner, y) \rightarrow y = \ulcorner \varphi \urcorner). \quad (2)$$

*inc:inp:fix:
repdiag2*

Now we show that $\mathbf{Q} \vdash \varphi \leftrightarrow \psi(\ulcorner \varphi \urcorner)$. We argue informally, using just logic and facts **derivável** in $\mathbf{Q}$.

First, suppose φ , O.s., $\alpha(\ulcorner \alpha(x) \urcorner)$. Going back to the definition of $\alpha(x)$, we see that $\alpha(\ulcorner \alpha(x) \urcorner)$ just is

$$\exists y (\theta_{\text{diag}}(\ulcorner \alpha(x) \urcorner, y) \wedge \psi(y)).$$

Consider such a y . Since $\theta_{\text{diag}}(\ulcorner \alpha(x) \urcorner, y)$, by **Eq. (2)**, $y = \ulcorner \varphi \urcorner$. So, from $\psi(y)$ we have $\psi(\ulcorner \varphi \urcorner)$.

Now suppose $\psi(\ulcorner \varphi \urcorner)$. By **Eq. (1)**, we have

$$\theta_{\text{diag}}(\ulcorner \alpha(x) \urcorner, \ulcorner \varphi \urcorner) \wedge \psi(\ulcorner \varphi \urcorner).$$

It follows that

$$\exists y (\theta_{\text{diag}}(\ulcorner \alpha(x) \urcorner, y) \wedge \psi(y)).$$

But that's just $\alpha(\ulcorner \alpha(x) \urcorner)$, O.s., φ . □

digression

You should compare this to the proof of the fixed-point lemma in computability theory. The difference is that here we want to define a *statement* in terms of itself, whereas there we wanted to define a *function* in terms of itself; this difference aside, it is really the same idea.

Problema inp.1. A **fórmula** $\varphi(x)$ is a *truth definition* if $\mathbf{Q} \vdash \psi \leftrightarrow \varphi(\ulcorner \psi \urcorner)$ for all **sentenças** ψ . Show that no **fórmula** is a truth definition by using the fixed-point lemma.

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Referências Bibliográficas