-NoValue-

art.1 Derivações in Natural Deduction

inc:art:pnd:sec In order to arithmetize derivações, we must represent derivações as numbers. explanation
Since derivações are trees of fórmulas where each inference carries one or two labels, a recursive representation is the most obvious approach: we represent a derivação as a tuple, the components of which are the number of immediate sub-derivações leading to the premises of the last inference, the representations of these sub-derivações, and the end-fórmula, the discharge label of the last inference, and a number indicating the type of the last inference.

Definição art.1. If δ is a derivação in natural deduction, then ${}^{\#}\delta^{\#}$ is defined inductively as follows:

1. If δ consists only of the assumption φ, then ${}^{\#}\delta^{\#}$ is $\langle 0, {}^{\#}\varphi^{\#}, n\rangle$. The number n is 0 if it is an não descarregada assumption, and the numerical label otherwise.

2. If δ ends in an inference with zero, one, two, or three premises, then ${}^{\#}\delta^{\#}$ is

$$\begin{aligned}&\langle 0, {}^{\#}\varphi^{\#}, n, k\rangle,\\ &\langle 1, {}^{\#}\delta_1{}^{\#}, {}^{\#}\varphi^{\#}, n, k\rangle,\\ &\langle 2, {}^{\#}\delta_1{}^{\#}, {}^{\#}\delta_2{}^{\#}, {}^{\#}\varphi^{\#}, n, k\rangle, \text{ or}\\ &\langle 3, {}^{\#}\delta_1{}^{\#}, {}^{\#}\delta_2{}^{\#}, {}^{\#}\delta_3{}^{\#}, {}^{\#}\varphi^{\#}, n, k\rangle,\end{aligned}$$

respectively. Here δ_1, δ_2, δ_3 are the sub-derivações ending in the premise(s) of the last inference in δ, φ is the conclusion of the last inference in δ, n is the discharge label of the last inference (0 if the inference does not discharge any assumptions), and k is given by the following table according to which rule was used in the last inference.

Rule:	$\land$Intro	$\land$Elim	$\lor$Intro	$\lor$Elim
k:	1	2	3	4
Rule:	$\to$Intro	$\to$Elim	$\lnot$Intro	$\lnot$Elim
k:	5	6	7	8
Rule:	$\bot_I$	$\bot_C$	$\forall$Intro	$\forall$Elim
k:	9	10	11	12
Rule:	$\exists$Intro	$\exists$Elim	$=$Intro	$=$Elim
k:	13	14	15	16

Exemplo art.2. Consider the very simple derivação

$$\dfrac{\dfrac{[\varphi \land \psi]^1}{\varphi}\land\text{Elim}}{(\varphi \land \psi) \to \varphi}\;{}_1\to\text{Intro}$$

The Gödel number of the assumption would be $d_0 = \langle 0, {}^{\#}\varphi \wedge \psi^{\#}, 1\rangle$. The Gödel number of the derivação ending in the conclusion of $\wedge$Elim would be $d_1 = \langle 1, d_0, {}^{\#}\varphi^{\#}, 0, 2\rangle$ (1 since $\wedge$Elim has one premise, the Gödel number of conclusion φ, 0 because no assumption is discharged, and 2 is the number coding $\wedge$Elim). The Gödel number of the entire derivação then is $\langle 1, d_1, {}^{\#}((\varphi \wedge \psi) \to \varphi)^{\#}, 1, 5\rangle$, O.s.,

$$\langle 1, \langle 1, \langle 0, {}^{\#}(\varphi \wedge \psi)^{\#}, 1\rangle, {}^{\#}\varphi^{\#}, 0, 2\rangle, {}^{\#}((\varphi \wedge \psi) \to \varphi)^{\#}, 1, 5\rangle.$$

explanation

Having settled on a representation of derivações, we must also show that we can manipulate Gödel numbers of such derivações primitive recursively, and express their essential properties and relations. Some operations are simple: p.ex., given a Gödel number d of a derivação, $\mathrm{EndFmla}(d) = (d)_{(d)_0+1}$ gives us the Gödel number of its end-fórmula, $\mathrm{DischargeLabel}(d) = (d)_{(d)_0+2}$ gives us the discharge label and $\mathrm{LastRule}(d) = (d)_{(d)_0+3}$ the number indicating the type of the last inference. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation "δ is a derivação of φ from Γ" is a primitive recursive relation of the Gödel numbers of δ and φ.

Proposição art.3. *The following relations are primitive recursive:*

1. *φ occurs as an assumption in δ with label n.*
2. *All assumptions in δ with label n are of the form φ (O.s., we can descarrega the assumption φ using label n in δ).*

Prova. We have to show that the corresponding relations between Gödel numbers of fórmulas and Gödel numbers of derivações are primitive recursive.

1. We want to show that $\mathrm{Assum}(x, d, n)$, which holds if x is the Gödel number of an assumption of the derivação with Gödel number d labelled n, is primitive recursive. This is the case if the derivação with Gödel number $\langle 0, x, n\rangle$ is a sub-derivação of d. Note that the way we code derivações is a special case of the coding of trees introduced in **??**, so the primitive recursive function $\mathrm{SubtreeSeq}(d)$ gives a sequence of Gödel numbers of all sub-derivações of d (of length at most d). So we can define

$$\mathrm{Assum}(x, d, n) \Leftrightarrow (\exists i < d)\ (\mathrm{SubtreeSeq}(d))_i = \langle 0, x, n\rangle.$$

2. We want to show that $\mathrm{Discharge}(x, d, n)$, which holds if all assumptions with label n in the derivação with Gödel number d all are the fórmula with Gödel number x. But this relation holds iff $(\forall y < d)\ (\mathrm{Assum}(y, d, n) \to y = x)$. □

Proposição art.4. *The property* $\mathrm{Correct}(d)$ *which holds iff the last inference in the derivação δ with Gödel number d is correct, is primitive recursive.* inc:art:pnd: prop:followsby

Prova. Here we have to show that for each rule of inference R the relation $\mathrm{FollowsBy}_R(d)$ is primitive recursive, where $\mathrm{FollowsBy}_R(d)$ holds iff d is the Gödel number of derivação δ, and the end-fórmula of δ follows by a correct application of R from the immediate sub-derivações of δ.

A simple case is that of the $\land$Intro rule. If δ ends in a correct $\land$Intro inference, it looks like this:

$$\dfrac{\begin{matrix}\vdots\,\delta_1 \\ \varphi\end{matrix} \qquad \begin{matrix}\vdots\,\delta_2 \\ \psi\end{matrix}}{\varphi \land \psi}\;\land\mathrm{Intro}$$

Then the Gödel number d of δ is $\langle 2, d_1, d_2, {}^{\#}(\varphi \land \psi)^{\#}, 0, k\rangle$ where $\mathrm{EndFmla}(d_1) = {}^{\#}\varphi^{\#}$, $\mathrm{EndFmla}(d_2) = {}^{\#}\psi^{\#}$, $n = 0$, and $k = 1$. So we can define $\mathrm{FollowsBy}_{\land\mathrm{Intro}}(d)$ as

$$\begin{aligned}&(d)_0 = 2 \land \mathrm{DischargeLabel}(d) = 0 \land \mathrm{LastRule}(d) = 1 \land \\ &\qquad\qquad \mathrm{EndFmla}(d) = \\ &\qquad\qquad {}^{\#}({}^{\#} \frown \mathrm{EndFmla}((d)_1) \frown {}^{\#}\land^{\#} \frown \mathrm{EndFmla}((d)_2) \frown {}^{\#})^{\#}.\end{aligned}$$

Another simple example is the $=$Intro rule. This has no premises, so $(d)_0 = 0$, like assumptions. It also has no discharge label, O.s., $n = 0$. However, φ must be of the form $t = t$, for a closed term t. Here, a primitive recursive definition is

$$\begin{aligned}&(d)_0 = 0 \land \mathrm{DischargeLabel}(d) = 0 \land \\ &\qquad (\exists t < d)\,(\mathrm{ClTerm}(t) \land \mathrm{EndFmla}(d) = \\ &\qquad\qquad {}^{\#}{=}({}^{\#} \frown t \frown {}^{\#},{}^{\#} \frown t \frown {}^{\#})^{\#}).\end{aligned}$$

For a more complicated example, $\mathrm{FollowsBy}_{\to\mathrm{Intro}}(d)$ holds iff the end-fórmula of δ is of the form $(\varphi \to \psi)$, where the end-fórmula of δ_1 is ψ, and any assumption in δ labelled n is of the form φ. We can express this primitive recursively by

$$\begin{aligned}&(d)_0 = 1 \land \\ &\qquad (\exists a < d)\,(\mathrm{Discharge}(a, (d)_1, \mathrm{DischargeLabel}(d)) \land \\ &\qquad \mathrm{EndFmla}(d) = ({}^{\#}({}^{\#} \frown a \frown {}^{\#}{\to}^{\#} \frown \mathrm{EndFmla}((d)_1) \frown {}^{\#})^{\#}))\end{aligned}$$

(Think of a as the Gödel number of φ).

For another example, consider $\exists$Intro. Here, the last inference in δ is correct iff there is a fórmula φ, a closed term t and a variável x such that $\varphi[t/x]$ is the end-fórmula of the derivação δ_1 and $\exists x\,\varphi$ is the conclusion of the last inference. So, $\mathrm{FollowsBy}_{\exists\mathrm{Intro}}(d)$ holds iff

$$\begin{aligned}&(d)_0 = 1 \land \mathrm{DischargeLabel}(d) = 0 \land \\ &\qquad (\exists a < d)\,(\exists x < d)\,(\exists t < d)\,(\mathrm{ClTerm}(t) \land \mathrm{Var}(x) \land \\ &\qquad \mathrm{Subst}(a, t, x) = \mathrm{EndFmla}((d)_1) \land \mathrm{EndFmla}(d) = ({}^{\#}\exists^{\#} \frown x \frown a)).\end{aligned}$$

We then define Correct(d) as

$$\begin{aligned}&\mathrm{Sent}(\mathrm{EndFmla}(d)) \land \\ &\qquad (\mathrm{LastRule}(d) = 1 \land \mathrm{FollowsBy}_{\land\mathrm{Intro}}(d)) \lor \cdots \lor \\ &\qquad\quad (\mathrm{LastRule}(d) = 16 \land \mathrm{FollowsBy}_{=\mathrm{Elim}}(d)) \lor \\ &\qquad\qquad\qquad (\exists n < d)\,(\exists x < d)\,(d = \langle 0, x, n\rangle).\end{aligned}$$

The first line ensures that the end-fórmula of d is a sentence. The last line covers the case where d is just an assumption. □

Problema art.1. Define the following properties as in Proposição art.4:

1. $\mathrm{FollowsBy}_{\to\mathrm{Elim}}(d)$,
2. $\mathrm{FollowsBy}_{=\mathrm{Elim}}(d)$,
3. $\mathrm{FollowsBy}_{\lor\mathrm{Elim}}(d)$,
4. $\mathrm{FollowsBy}_{\forall\mathrm{Intro}}(d)$.

For the last one, you will have to also show that you can test primitive recursively if the last inference of the derivação with Gödel number d satisfies the eigenvariable condition, O.s., the eigenvariable a of the ∀Intro inference occurs neither in the end-fórmula of d nor in an open assumption of d. You may use the primitive recursive predicate OpenAssum from Proposição art.6 for this.

Proposição art.5. *The relation* Deriv(d) *which holds if d is the Gödel number of a correct derivação δ, is primitive recursive.* inc:art:pnd: prop:deriv

Prova. A derivação δ is correct if every one of its inferences is a correct application of a rule, O.s., if every one of its sub-derivações ends in a correct inference. So, Deriv(d) iff

$$(\forall i < \mathrm{len}(\mathrm{SubtreeSeq}(d)))\ \mathrm{Correct}((\mathrm{SubtreeSeq}(d))_i)$$ □

Proposição art.6. *The relation* OpenAssum(z, d) *that holds if z is the Gödel number of a não descarregada assumption φ of the derivação δ with Gödel number d, is primitive recursive.* inc:art:pnd: prop:openassum

Prova. An occurrence of an assumption is descarregada if it occurs with label n in a sub-derivação of δ that ends in a rule with discharge label n. So φ is a não descarregada assumption of δ if at least one of its occurrences is not descarregada in δ. We must be careful: δ may contain both descarregada and não descarregada occurrences of φ.

Consider a sequence $\delta_0, \ldots, \delta_k$ where $\delta_0 = \delta$, δ_k is the assumption $[\varphi]^n$ (for some n), and δ_{i+1} is an immediate sub-derivação of δ_i. If such a sequence exists in which no δ_i ends in an inference with discharge label n, then φ is a não descarregada assumption of δ.

The primitive recursive function $\mathrm{SubtreeSeq}(d)$ provides us with a sequence of Gödel numbers of all sub-derivações of δ. Any sequence of Gödel numbers of sub-derivações of δ is a subsequence of it. Being a subsequence of is a primitive recursive relation: $\mathrm{Subseq}(s, s')$ holds iff $(\forall i < \mathrm{len}(s))\ \exists j < \mathrm{len}(s')\ (s)_i = (s')_j$. Being an immediate sub-derivação is as well: $\mathrm{Subderiv}(d, d')$ iff $(\exists j < (d')_0)\ d = (d')_j$. So we can define $\mathrm{OpenAssum}(z, d)$ by

$$\begin{aligned}(\exists s < \mathrm{SubtreeSeq}(d))\ (\mathrm{Subseq}(s, \mathrm{SubtreeSeq}(d)) \land (s)_0 = d \land\\ (\exists n < d)\ ((s)_{\mathrm{len}(s) \dot{-} 1} = \langle 0, z, n\rangle \land\\ (\forall i < (\mathrm{len}(s) \dot{-} 1))\ (\mathrm{Subderiv}((s)_{i+1}, (s)_i) \land\\ \mathrm{DischargeLabel}((s)_i) \neq n))).\end{aligned}$$

□

inc:art:pnd: prop:prf-prim-rec

Proposição art.7. *Suppose Γ is a primitive recursive set of sentenças. Then the relation $\mathrm{Prf}_\Gamma(x, y)$ expressing "x is the code of a derivação δ of φ from não descarregada assumptions in Γ and y is the Gödel number of φ" is primitive recursive.*

Prova. Suppose "$y \in \Gamma$" is given by the primitive recursive predicate $R_\Gamma(y)$. We have to show that $\mathrm{Prf}_\Gamma(x, y)$ which holds iff y is the Gödel number of a sentence φ and x is the code of a natural deduction derivação with end fórmula φ and all não descarregada assumptions in Γ is primitive recursive.

By Proposição art.5, the property $\mathrm{Deriv}(x)$ which holds iff x is the Gödel number of a correct derivação δ in natural deduction is primitive recursive. Thus we can define $\mathrm{Prf}_\Gamma(x, y)$ by

$$\begin{aligned}\mathrm{Prf}_\Gamma(x, y) \Leftrightarrow{}& \mathrm{Deriv}(x) \land \mathrm{EndFmla}(x) = y \land\\ & (\forall z < x)\ (\mathrm{OpenAssum}(z, x) \to R_\Gamma(z)).\end{aligned}$$

□

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Referências Bibliográficas