

-NoValue-

tab.1 Soundness

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sec A **derivação** system, such as tableaux, is *sound* if it cannot **deriva** things that explanation do not actually hold. Soundness is thus a kind of guaranteed safety property for **derivação** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

1. every **derivável** φ is valid;
2. if a **sentença** is **derivável** from some others, it is also a consequence of them;
3. if a set of **sentenças** is inconsistent, it is unsatisfiable.

These are important properties of a **derivação** system. If any of them do not hold, the **derivação** system is deficient—it would **deriva** too much. Consequently, establishing the soundness of a **derivação** system is of the utmost importance.

Because all these proof-theoretic properties are defined via closed **tableaux** of some kind or other, proving (1)–(3) above requires proving something about the semantic properties of closed **tableaux**. We will first define what it means for a **fórmula assinada** to be satisfied in a structure, and then show that if a **tableau** is closed, no structure satisfies all its assumptions. (1)–(3) then follow as corollaries from this result.

Definição tab.1. A **estrutura** $\mathfrak{M}$ *satisfies* a **fórmula assinada** $\mathbb{T}\varphi$ iff $\mathfrak{M} \models \varphi$, and it satisfies $\mathbb{F}\varphi$ iff $\mathfrak{M} \not\models \varphi$. $\mathfrak{M}$ satisfies a set of **fórmulas assinadas** Γ iff it satisfies every $S\varphi \in \Gamma$. Γ is *satisfiable* if there is a **estrutura** that satisfies it, and *unsatisfiable* otherwise.

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thm:tableau-soundness **Teorema tab.2 (Soundness).** *If Γ has a closed **tableau**, Γ is unsatisfiable.*

Prova. Let's call a branch of a **tableau** *satisfiable* iff the set of **fórmulas assinadas** on it is satisfiable, and let's call a **tableau** *satisfiable* if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable **tableau** by one of the rules of inference always results in a satisfiable **tableau**. This will prove the theorem: any closed **tableau** results by applying rules of inference to the **tableau** consisting only of assumptions from Γ . So if Γ were satisfiable, any **tableau** for it would be satisfiable. A closed **tableau**, however, is clearly not satisfiable: every branch contains both $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$, and no structure can both satisfy and not satisfy φ .

Suppose we have a satisfiable **tableau**, O.s., a **tableau** with at least one satisfiable branch. Applying a rule of inference either adds **fórmulas assinadas** to a branch, or splits a branch in two. If the **tableau** has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable

branch in the extended **tableau**, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let Γ be the set of **fórmulas assinadas** on that branch, and let $S\varphi \in \Gamma$ be the **fórmula assinada** to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, O.s., Γ together with the conclusions of the rule, is still satisfiable. If the rule results in a split branch, we have to show that at least one of the two resulting branches is satisfiable.

First, we consider the possible inferences that do not result in a split branch.

1. The branch is expanded by applying $\neg\mathbb{T}$ to $\mathbb{T}\neg\psi \in \Gamma$. Then the extended branch contains the **fórmulas assinadas** $\Gamma \cup \{\mathbb{F}\psi\}$. Suppose $\mathfrak{M} \models \Gamma$. In particular, $\mathfrak{M} \models \neg\psi$. Thus, $\mathfrak{M} \not\models \psi$, O.s., $\mathfrak{M}$ satisfies $\mathbb{F}\psi$.
2. The branch is expanded by applying $\neg\mathbb{F}$ to $\mathbb{F}\neg\psi \in \Gamma$: Exercise.
3. The branch is expanded by applying $\wedge\mathbb{T}$ to $\mathbb{T}\psi \wedge \chi \in \Gamma$, which results in two new **fórmulas assinadas** on the branch: $\mathbb{T}\psi$ and $\mathbb{T}\chi$. Suppose $\mathfrak{M} \models \Gamma$, in particular $\mathfrak{M} \models \psi \wedge \chi$. Then $\mathfrak{M} \models \psi$ and $\mathfrak{M} \models \chi$. This means that $\mathfrak{M}$ satisfies both $\mathbb{T}\psi$ and $\mathbb{T}\chi$.
4. The branch is expanded by applying $\vee\mathbb{F}$ to $\mathbb{F}\psi \vee \chi \in \Gamma$: Exercise.
5. The branch is expanded by applying $\rightarrow\mathbb{F}$ to $\mathbb{F}\psi \rightarrow \chi \in \Gamma$: This results in two new **fórmulas assinadas** on the branch: $\mathbb{T}\psi$ and $\mathbb{F}\chi$. Suppose $\mathfrak{M} \models \Gamma$, in particular $\mathfrak{M} \not\models \psi \rightarrow \chi$. Then $\mathfrak{M} \models \psi$ and $\mathfrak{M} \not\models \chi$. This means that $\mathfrak{M}$ satisfies both $\mathbb{T}\psi$ and $\mathbb{F}\chi$.
6. The branch is expanded by applying $\forall\mathbb{T}$ to $\mathbb{T}\forall x\psi(x) \in \Gamma$: This results in a new **fórmula assinada** $\mathbb{T}\varphi(t)$ on the branch. Suppose $\mathfrak{M} \models \Gamma$, in particular, $\mathfrak{M} \models \forall x\varphi(x)$. By ??, $\mathfrak{M} \models \varphi(t)$. Consequently, $\mathfrak{M}$ satisfies $\mathbb{T}\varphi(t)$.
7. The branch is expanded by applying $\forall\mathbb{F}$ to $\mathbb{F}\forall x\psi(x) \in \Gamma$: This results in a new **fórmula assinada** $\mathbb{F}\varphi(a)$ where a is a **símbolo de constante** not occurring in Γ . Since Γ is satisfiable, there is a $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma$, in particular $\mathfrak{M} \not\models \forall x\psi(x)$. We have to show that $\Gamma \cup \{\mathbb{F}\varphi(a)\}$ is satisfiable. To do this, we define a suitable $\mathfrak{M}'$ as follows.
By ??, $\mathfrak{M} \not\models \forall x\psi(x)$ iff for some s , $\mathfrak{M}, s \not\models \psi(x)$. Now let $\mathfrak{M}'$ be just like $\mathfrak{M}$, except $a^{\mathfrak{M}'} = s(x)$. By ??, for any $\mathbb{T}\chi \in \Gamma$, $\mathfrak{M}' \models \chi$, and for any $\mathbb{F}\chi \in \Gamma$, $\mathfrak{M}' \not\models \chi$, since a does not occur in Γ .
By ??, $\mathfrak{M}', s \not\models \varphi(x)$. By ??, $\mathfrak{M}', s \not\models \varphi(a)$. Since $\varphi(a)$ is a **sentença**, by ??, $\mathfrak{M}' \not\models \varphi(a)$, O.s., $\mathfrak{M}'$ satisfies $\mathbb{F}\varphi(a)$.
8. The branch is expanded by applying $\exists\mathbb{T}$ to $\mathbb{T}\exists x\psi(x) \in \Gamma$: Exercise.
9. The branch is expanded by applying $\exists\mathbb{F}$ to $\mathbb{F}\exists x\psi(x) \in \Gamma$: Exercise.

Now let's consider the possible inferences that result in a split branch.

1. The branch is expanded by applying $\wedge\mathbb{F}$ to $\mathbb{F}\psi \wedge \chi \in \Gamma$, which results in two branches, a left one continuing through $\mathbb{F}\psi$ and a right one through $\mathbb{F}\chi$. Suppose $\mathfrak{M} \models \Gamma$, in particular $\mathfrak{M} \models \psi \wedge \chi$. Then $\mathfrak{M} \models \psi$ or $\mathfrak{M} \models \chi$. In the former case, $\mathfrak{M}$ satisfies $\mathbb{F}\psi$, O.s., $\mathfrak{M}$ satisfies the formulas on the left branch. In the latter, $\mathfrak{M}$ satisfies $\mathbb{F}\chi$, O.s., $\mathfrak{M}$ satisfies the formulas on the right branch.
2. The branch is expanded by applying $\vee\mathbb{T}$ to $\mathbb{T}\psi \vee \chi \in \Gamma$: Exercise.
3. The branch is expanded by applying $\rightarrow\mathbb{T}$ to $\mathbb{T}\psi \rightarrow \chi \in \Gamma$: Exercise.
4. The branch is expanded by Cut: This results in two branches, one containing $\mathbb{T}\psi$, the other containing $\mathbb{F}\psi$. Since $\mathfrak{M} \models \Gamma$ and either $\mathfrak{M} \models \psi$ or $\mathfrak{M} \models \neg\psi$, $\mathfrak{M}$ satisfies either the left or the right branch. $\square$

Problema tab.1. Complete the proof of **Teorema tab.2**.

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cor:weak-soundness

Corolário tab.3. *If $\vdash \varphi$ then φ is valid.*

fol:tab:sou:

cor:entailment-soundness

Corolário tab.4. *If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*

Prova. If $\Gamma \vdash \varphi$ then for some $\psi_1, \dots, \psi_n \in \Gamma$, $\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$ has a closed **tableau**. By **Teorema tab.2**, every **estrutura** $\mathfrak{M}$ either makes some ψ_i false or makes φ true. Hence, if $\mathfrak{M} \models \Gamma$ then also $\mathfrak{M} \models \varphi$. $\square$

fol:tab:sou:

cor:consistency-soundness

Corolário tab.5. *If Γ is satisfiable, then it is consistent.*

Prova. We prove the contrapositive. Suppose that Γ is not consistent. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ and a closed **tableau** for $\{\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$. By **Teorema tab.2**, there is no $\mathfrak{M}$ such that $\mathfrak{M} \models \psi_i$ for all $i = 1, \dots, n$. But then Γ is not satisfiable. $\square$

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Referências Bibliográficas