-NoValue-

axd.1 Soundness

fol:axd:sou: sec

A derivação system, such as axiomatic deduction, is *sound* if it cannot deriva things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivação systems. Depending on which proof theoretic property is in question, we would like to know for instance, that explanation

1. every derivável φ is valid;
2. if φ is derivável from some others Γ, it is also a consequence of them;
3. if a set of fórmulas Γ is inconsistent, it is unsatisfiable.

These are important properties of a derivação system. If any of them do not hold, the derivação system is deficient—it would deriva too much. Consequently, establishing the soundness of a derivação system is of the utmost importance.

Proposição axd.1. *If φ is an axiom, then $\mathfrak{M}, s \vDash \varphi$ for each estrutura $\mathfrak{M}$ and assignment s.*

Prova. We have to verify that all the axioms are valid. For instance, here is the case for **??**: suppose t is livre para x in φ, and assume $\mathfrak{M}, s \vDash \forall x\, \varphi$. Then by definition of satisfaction, for each $s' \sim_x s$, also $\mathfrak{M}, s' \vDash \varphi$, and in particular this holds when $s'(x) = \mathrm{Val}^{\mathfrak{M}}_s(t)$. By **??**, $\mathfrak{M}, s \vDash \varphi[t/x]$. This shows that $\mathfrak{M}, s \vDash (\forall x\, \varphi \to \varphi[t/x])$. □

fol:axd:sou: thm:soundness

Teorema axd.2 (Soundness). *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

Prova. By induction on the length of the derivação of φ from Γ. If there are no steps justified by inferences, then all fórmulas in the derivação are either instances of axioms or are in Γ. By the previous proposition, all the axioms are valid, and hence if φ is an axiom then $\Gamma \vDash \varphi$. If $\varphi \in \Gamma$, then trivially $\Gamma \vDash \varphi$.

If the last step of the derivação of φ is justified by modus ponens, then there are fórmulas ψ and $\psi \to \varphi$ in the derivação, and the induction hypothesis applies to the part of the derivação ending in those fórmulas (since they contain at least one fewer step justified by an inference). So, by induction hypothesis, $\Gamma \vDash \psi$ and $\Gamma \vDash \psi \to \varphi$. Then $\Gamma \vDash \varphi$ by **??**.

Now suppose the last step is justified by QR. Then that step has the form $\chi \to \forall x\, B(x)$ and there is a preceding step $\chi \to \psi(c)$ with c not in Γ, χ, or $\forall x\, B(x)$. By induction hypothesis, $\Gamma \vDash \chi \to \psi(c)$. By **??**, $\Gamma \cup \{\chi\} \vDash \psi(c)$.

Consider some structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash \Gamma \cup \{\chi\}$. We need to show that $\mathfrak{M} \vDash \forall x\, \psi(x)$. Since $\forall x\, \psi(x)$ is a sentença, this means we have to show that for every variable assignment s, $\mathfrak{M}, s \vDash \psi(x)$ (**??**). Since $\Gamma \cup \{\chi\}$ consists entirely of sentences, $\mathfrak{M}, s \vDash \theta$ for all $\theta \in \Gamma$ by **??**. Let $\mathfrak{M}'$ be like $\mathfrak{M}$ except that $c^{\mathfrak{M}'} = s(x)$. Since c does not occur in Γ or χ, $\mathfrak{M}' \vDash \Gamma \cup \{\chi\}$ by **??**. Since

$\Gamma \cup \{\chi\} \vDash \psi(c)$, $\mathfrak{M}' \vDash B(c)$. Since $\psi(c)$ is a sentença, $\mathfrak{M}', s \vDash \psi(c)$ by **??**. $\mathfrak{M}', s \vDash \psi(x)$ iff $\mathfrak{M}', s \vDash \psi(c)$ by **??** (recall that $\psi(c)$ is just $\psi(x)[c/x]$). So, $\mathfrak{M}', s \vDash \psi(x)$. Since c does not occur in $\psi(x)$, by **??**, $\mathfrak{M}, s \vDash \psi(x)$. But s was an arbitrary variable assignment, so $\mathfrak{M} \vDash \forall x\, \psi(x)$. Thus $\Gamma \cup \{\chi\} \vDash \forall x\, \psi(x)$. By **??**, $\Gamma \vDash \chi \rightarrow \forall x\, \psi(x)$.

The case where φ is justified by QR but is of the form $\exists x\, \psi(x) \rightarrow \chi$ is left as an exercise. $\square$

Problema axd.1. Complete the proof of Teorema axd.2.

Corolário axd.3. *If $\vdash \varphi$, then φ is valid.* *fol:axd:sou:cor:weak-soundness*

Corolário axd.4. *If Γ is satisfiable, then it is consistent.* *fol:axd:sou:cor:consistency-soundness*

Prova. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \bot$, O.s., there is a derivação of $\bot$ from Γ. By Teorema axd.2, any estrutura $\mathfrak{M}$ that satisfies Γ must satisfy $\bot$. Since $\mathfrak{M} \nvDash \bot$ for every estrutura $\mathfrak{M}$, no $\mathfrak{M}$ can satisfy Γ, O.s., Γ is not satisfiable. $\square$

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Referências Bibliográficas