Capítulo udf

Axiomatic Derivações

> No effort has been made yet to ensure that the material in this chapter respects various tags indicating which connectives and quantifiers are primitive or defined: all are assumed to be primitive, except $\leftrightarrow$ which is assumed to be defined. If the FOL tag is true, we produce a version with quantifiers, otherwise without.

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axd.1 Rules and Derivações

fol:axd:rul:sec Axiomatic derivações are perhaps the simplest derivação system for logic. A derivação is just a sequence of fórmulas. To count as a derivação, every fórmula in the sequence must either be an instance of an axiom, or must follow from one or more fórmulas that precede it in the sequence by a rule of inference. A derivação deriva its last fórmula. explanation

Definição axd.1 (Derivabilidade). If Γ is a set of fórmulas of $\mathcal{L}$ then a *derivação* from Γ is a finite sequence $\varphi_1, \dots, \varphi_n$ of fórmulas where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. φ_i follows from some φ_j (and φ_k) with $j < i$ (and $k < i$) by a rule of inference.

What counts as a correct derivação depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step A_i in a derivação is a correct inference step.

Definição axd.2 (Rule of inference). A *rule of inference* gives a sufficient condition for what counts as a correct inference step in a derivação from Γ.

For instance, since any one-element sequence φ with $\varphi \in \Gamma$ trivially counts as a derivação, the following might be a very simple rule of inference:

> If $\varphi \in \Gamma$, then φ is always a correct inference step in any derivação from Γ.

Similarly, if φ is one of the axioms, then φ by itself is a derivação, and so this is also a rule of inference:

> If φ is an axiom, then φ is a correct inference step.

It gets more interesting if the rule of inference appeals to fórmulas that appear before the step considered. The following rule is called *modus ponens:*

> If $\psi \rightarrow \varphi$ and ψ occur higher up in the derivação, then φ is a correct inference step.

If this is the only rule of inference, then our definition of derivação above amounts to this: $\varphi_1, \ldots, \varphi_n$ is a derivação iff for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. for some $j < i$, φ_j is $\psi \rightarrow \varphi_i$, and for some $k < i$, φ_k is ψ.

The last clause says that φ_i follows from φ_j ($\psi \rightarrow \varphi_i$) and φ_k (ψ) by modus ponens. If we can go from 1 to n, and each time we find a fórmula φ_i that is either in Γ, an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct derivação.

Definição axd.3 (Derivabilidade). A fórmula φ is *derivável* from Γ, written $\Gamma \vdash \varphi$, if there is a derivação from Γ ending in φ.

Definição axd.4 (Theorems). A fórmula φ is a *theorem* if there is a derivação of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

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axd.2 Axioms and Rules for the Propositional Connectives

fol:axd:prp:sec

Definição axd.5 (Axioms). The set of Ax_0 of *axioms* for the propositional connectives comprises all fórmulas of the following forms:

fol:axd:prp:ax:land1
$$(\varphi \wedge \psi) \to \varphi \tag{axd.1}$$
fol:axd:prp:ax:land2
$$(\varphi \wedge \psi) \to \psi \tag{axd.2}$$
fol:axd:prp:ax:land3
$$\varphi \to (\psi \to (\varphi \wedge \psi)) \tag{axd.3}$$
fol:axd:prp:ax:lor1
$$\varphi \to (\varphi \vee \psi) \tag{axd.4}$$
fol:axd:prp:ax:lor2
$$\varphi \to (\psi \vee \varphi) \tag{axd.5}$$
fol:axd:prp:ax:lor3
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)) \tag{axd.6}$$
fol:axd:prp:ax:lif1
$$\varphi \to (\psi \to \varphi) \tag{axd.7}$$
fol:axd:prp:ax:lif2
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \tag{axd.8}$$
fol:axd:prp:ax:lnot1
$$(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi) \tag{axd.9}$$
fol:axd:prp:ax:lnot2
$$\neg\varphi \to (\varphi \to \psi) \tag{axd.10}$$
fol:axd:prp:ax:ltrue
$$\top \tag{axd.11}$$
fol:axd:prp:ax:lfalse1
$$\bot \to \varphi \tag{axd.12}$$
fol:axd:prp:ax:lfalse2
$$(\varphi \to \bot) \to \neg\varphi \tag{axd.13}$$
fol:axd:prp:ax:dne
$$\neg\neg\varphi \to \varphi \tag{axd.14}$$

Definição axd.6 (Modus ponens). If ψ and $\psi \to \varphi$ already occur in a derivação, then φ is a correct inference step.

We'll abbreviate the rule modus ponens as "MP."
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axd.3 Axioms and Rules for Quantifiers

fol:axd:qua:sec

Definição axd.7 (Axioms for quantifiers). The *axioms* governing quantifiers are all instances of the following:

fol:axd:qua:ax:q1
$$\forall x\, \psi \to \psi(t), \tag{axd.15}$$
fol:axd:qua:ax:q2
$$\psi(t) \to \exists x\, \psi. \tag{axd.16}$$

for any closed term t.

Definição axd.8 (Rules for quantifiers).

If $\psi \to \varphi(a)$ already occurs in the derivação and a does not occur in Γ or ψ, then $\psi \to \forall x\, \varphi(x)$ is a correct inference step.

If $\varphi(a) \rightarrow \psi$ already occurs in the derivação and a does not occur in Γ or ψ, then $\exists x\, \varphi(x) \rightarrow \psi$ is a correct inference step.

We'll abbreviate either of these by "QR."
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axd.4 Examples of Derivações

fol:axd:pro:sec

Exemplo axd.9. Suppose we want to prove $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$. Clearly, this is not an instance of any of our axioms, so we have to use the MP rule to deriva it. Our only rule is MP, which given φ and $\varphi \rightarrow \psi$ allows us to justify ψ. One strategy would be to use Eq. (axd.6) with φ being $\neg\theta$, ψ being α, and χ being $\theta \rightarrow \alpha$, O.s., the instance

$$(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha))).$$

Why? Two applications of MP yield the last part, which is what we want. And we easily see that $\neg\theta \rightarrow (\theta \rightarrow \alpha)$ is an instance of Eq. (axd.10), and $\alpha \rightarrow (\theta \rightarrow \alpha)$ is an instance of Eq. (axd.7). So our derivação is:

1.	$\neg\theta \rightarrow (\theta \rightarrow \alpha)$	Eq. (axd.10)
2.	$(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow$ $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$	Eq. (axd.6)
3.	$(\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha))$	1, 2, MP
4.	$\alpha \rightarrow (\theta \rightarrow \alpha)$	Eq. (axd.7)
5.	$(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$	3, 4, MP

Exemplo axd.10. Let's try to find a derivação of $\theta \rightarrow \theta$. It is not an instance of an axiom, so we have to use MP to deriva it. Eq. (axd.7) is an axiom of the form $\varphi \rightarrow \psi$ to which we could apply MP. To be useful, of course, the ψ which MP would justify as a correct step in this case would have to be $\theta \rightarrow \theta$, since this is what we want to deriva. That means φ would also have to be θ, O.s., we might look at this instance of Eq. (axd.7): fol:axd:pro:ex:identity

$$\theta \rightarrow (\theta \rightarrow \theta)$$

In order to apply MP, we would also need to justify the corresponding second premise, namely φ. But in our case, that would be θ, and we won't be able to deriva θ by itself. So we need a different strategy.

The other axiom involving just $\rightarrow$ is Eq. (axd.8), O.s.,

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

We could get to the last nested conditional by applying MP twice. Again, that would mean that we want an instance of Eq. (axd.8) where $\varphi \rightarrow \chi$ is $\theta \rightarrow \theta$, the fórmula we are aiming for. Then of course, φ and χ are both θ. How should we pick ψ so that both $\varphi \to (\psi \to \chi)$ and $\varphi \to \psi$, O.s., in our case $\theta \to (\psi \to \theta)$ and $\theta \to \psi$, are also derivável? Well, the first of these is already an instance of Eq. (axd.7), whatever we decide ψ to be. And $\theta \to \psi$ would be another instance of Eq. (axd.7) if ψ were $(\theta \to \theta)$. So, our derivação is:

1.	$\theta \to ((\theta \to \theta) \to \theta)$	Eq. (axd.7)
2.	$(\theta \to ((\theta \to \theta) \to \theta)) \to$ $((\theta \to (\theta \to \theta)) \to (\theta \to \theta))$	Eq. (axd.8)
3.	$(\theta \to (\theta \to \theta)) \to (\theta \to \theta)$	1, 2, MP
4.	$\theta \to (\theta \to \theta)$	Eq. (axd.7)
5.	$\theta \to \theta$	3, 4, MP

fol:axd:pro: ex:chain

Exemplo axd.11. Sometimes we want to show that there is a derivação of some fórmula from some other fórmulas Γ. For instance, let's show that we can deriva $\varphi \to \chi$ from $\Gamma = \{\varphi \to \psi, \psi \to \chi\}$.

1.	$\varphi \to \psi$	HYP
2.	$\psi \to \chi$	HYP
3.	$(\psi \to \chi) \to (\varphi \to (\psi \to \chi))$	Eq. (axd.7)
4.	$\varphi \to (\psi \to \chi)$	2, 3, MP
5.	$(\varphi \to (\psi \to \chi)) \to$ $((\varphi \to \psi) \to (\varphi \to \chi))$	Eq. (axd.8)
6.	$((\varphi \to \psi) \to (\varphi \to \chi))$	4, 5, MP
7.	$\varphi \to \chi$	1, 6, MP

The lines labelled "HYP" (for "hypothesis") indicate that the fórmula on that line is um membro of Γ.

fol:axd:pro: prop:chain

Proposição axd.12. *If $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \chi$, then $\Gamma \vdash \varphi \to \chi$*

Prova. Suppose $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \chi$. Then there is a derivação of $\varphi \to \psi$ from Γ; and a derivação of $\psi \to \chi$ from Γ as well. Combine these into a single derivação by concatenating them. Now add lines 3–7 of the derivação in the preceding example. This is a derivação of $\varphi \to \chi$—which is the last line of the new derivação—from Γ. Note that the justifications of lines 4 and 7 remain valid if the reference to line number 1 is replaced by reference to the last line of the derivação of $\varphi \to \psi$, and reference to line number 2 by reference to the last line of the derivação of $\psi \to \chi$. □

Problema axd.1. Show that the following hold by exhibiting derivações from the axioms:

1. $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
2. $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
3. $\neg(\varphi \vee \psi) \to \neg\varphi$

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axd.5 Derivações with Quantifiers

fol:axd:prq:sec

Exemplo axd.13. Let us give a derivação of $(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \forall x\, (\varphi(x) \land \psi(x))$.

First, note that

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \forall x\, \varphi(x)$$

is an instance of Eq. (axd.1), and

$$\forall x\, \varphi(x) \to \varphi(a)$$

of Eq. (axd.15). So, by Proposição axd.12, we know that

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \varphi(a)$$

is derivável. Likewise, since

$$\begin{aligned}(\forall x\, \varphi(x) \land \forall y\, \psi(y)) &\to \forall y\, \psi(y) \qquad \text{and}\\ \forall y\, \psi(y) &\to \psi(a)\end{aligned}$$

are instances of Eq. (axd.2) and Eq. (axd.15), respectively,

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \psi(a)$$

is derivable by Proposição axd.12. Using an appropriate instance of Eq. (axd.3) and two applications of MP, we see that

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to (\varphi(a) \land \psi(a))$$

is derivable. We can now apply QR to obtain

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \forall x\, (\varphi(x) \land \psi(x)).$$

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axd.6 Proof-Theoretic Notions

explanation Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentenças in estruturas, but by appeal to the derivabilidade or não derivabilidade of certain formulas. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*. fol:axd:ptn:sec

Definição axd.14 (Derivabilidade). A fórmula φ is *derivável* from Γ, written $\Gamma \vdash \varphi$, if there is a derivação from Γ ending in φ.

Definição axd.15 (Theorems). A fórmula φ is a *theorem* if there is a derivação of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definição axd.16 (Consistency). A set Γ of fórmulas is *consistent* if and only if $\Gamma \nvdash \bot$; it is *inconsistent* otherwise.

fol:axd:ptn:prop:reflexivity **Proposição axd.17 (Reflexivity).** *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Prova. The fórmula φ by itself is a derivação of φ from Γ. □

fol:axd:ptn:prop:monotonicity **Proposição axd.18 (Monotonicity).** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Prova. Any derivação of φ from Γ is also a derivação of φ from Δ. □

fol:axd:ptn:prop:transitivity **Proposição axd.19 (Transitivity).** *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Prova. Suppose $\{\varphi\} \cup \Delta \vdash \psi$. Then there is a derivação $\psi_1, \ldots, \psi_l = \psi$ from $\{\varphi\} \cup \Delta$. Some of the steps in that derivação will be correct because of a rule which refers to a prior line $\psi_i = \varphi$. By hypothesis, there is a derivação of φ from Γ, O.s., a derivação $\varphi_1, \ldots, \varphi_k = \varphi$ where every φ_i is an axiom, um membro of Γ, or correct by a rule of inference. Now consider the sequence

$$\varphi_1, \ldots, \varphi_k = \varphi, \psi_1, \ldots, \psi_l = \psi.$$

This is a correct derivação of ψ from $\Gamma \cup \Delta$ since every $B_i = \varphi$ is now justified by the same rule which justifies $\varphi_k = \varphi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \ldots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i, then $\Gamma \vdash \psi$.

fol:axd:ptn:prop:incons **Proposição axd.20.** *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every φ.*

Prova. Exercise. □

Problema axd.2. Prove Proposição axd.20.

fol:axd:ptn:prop:proves-compact **Proposição axd.21 (Compactness).**

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*

2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Prova. 1. If $\Gamma \vdash \varphi$, then there is a finite sequence of fórmulas $\varphi_1, \dots, \varphi_n$ so that $\varphi \equiv \varphi_n$ and each φ_i is either a logical axiom, um membro of Γ or follows from previous fórmulas by modus ponens. Take Γ_0 to be those φ_i which are in Γ. Then the derivação is likewise a derivação from Γ_0, and so $\Gamma_0 \vdash \varphi$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \bot$. □

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axd.7 The Deduction Theorem

As we've seen, giving derivações in an axiomatic system is cumbersome, and derivações may be hard to find. Rather than actually write out long lists of fórmulas, it is generally easier to argue that such derivações exist, by making use of a few simple results. We've already established three such results: Proposição axd.17 says we can always assert that $\Gamma \vdash \varphi$ when we know that $\varphi \in \Gamma$. Proposição axd.18 says that if $\Gamma \vdash \varphi$ then also $\Gamma \cup \{\psi\} \vdash \varphi$. And Proposição axd.19 implies that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. Here's another simple result, a "meta"-version of modus ponens: fol:axd:ded:sec

Proposição axd.22. *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.* *fol:axd:ded:prop:mp*

Prova. We have that $\{\varphi, \varphi \to \psi\} \vdash \psi$:

1.	φ	Hyp.
2.	$\varphi \to \psi$	Hyp.
3.	ψ	1, 2, MP

By Proposição axd.19, $\Gamma \vdash \psi$. □

The most important result we'll use in this context is the deduction theorem:

Teorema axd.23 (Deduction Theorem). *$\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \to \psi$.* *fol:axd:ded:thm:deduction-thm*

Prova. The "if" direction is immediate. If $\Gamma \vdash \varphi \to \psi$ then also $\Gamma \cup \{\varphi\} \vdash \varphi \to \psi$ by Proposição axd.18. Also, $\Gamma \cup \{\varphi\} \vdash \varphi$ by Proposição axd.17. So, by Proposição axd.22, $\Gamma \cup \{\varphi\} \vdash \psi$.

For the "only if" direction, we proceed by induction on the length of the derivação of ψ from $\Gamma \cup \{\varphi\}$.

For the induction basis, we prove the claim for every derivação of length 1. A derivação of ψ from $\Gamma \cup \{\varphi\}$ of length 1 consists of ψ by itself; and if it is correct ψ is either $\in \Gamma \cup \{\varphi\}$ or is an axiom. If $\psi \in \Gamma$ or is an axiom, then $\Gamma \vdash \psi$. We also have that $\Gamma \vdash \psi \to (\varphi \to \psi)$ by Eq. (axd.7), and Proposição axd.22 gives $\Gamma \vdash \varphi \to \psi$. If $\psi \in \{\varphi\}$ then $\Gamma \vdash \varphi \to \psi$ because the last sentença $\varphi \to \psi$ is the same as $\varphi \to \varphi$, and we have derivad that in Exemplo axd.10.

For the inductive step, suppose a derivação of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by modus ponens. (If it is not justified by modus ponens, $\psi \in \Gamma$, $\psi \equiv \varphi$, or ψ is an axiom, and the same reasoning as in the induction basis applies.) Then some previous steps in the derivação are $\chi \to \psi$ and χ, for some fórmula χ, O.s., $\Gamma \cup \{\varphi\} \vdash \chi \to \psi$ and $\Gamma \cup \{\varphi\} \vdash \chi$, and the respective derivações are shorter, so the inductive hypothesis applies to them. We thus have both:

$$\Gamma \vdash \varphi \to (\chi \to \psi);$$
$$\Gamma \vdash \varphi \to \chi.$$

But also

$$\Gamma \vdash (\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi)),$$

by Eq. (axd.8), and two applications of Proposição axd.22 give $\Gamma \vdash \varphi \to \psi$, as required. □

Notice how Eq. (axd.7) and Eq. (axd.8) were chosen precisely so that the Deduction Theorem would hold.

The following are some useful facts about derivabilidade, which we leave as exercises.

fol:axd:ded: prop:derivfacts **Proposição axd.24.**

fol:axd:ded: derivfacts:a 1. $\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;

fol:axd:ded: derivfacts:b 2. *If* $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$ *then* $\Gamma \cup \{\psi\} \vdash \varphi$ *(Contraposition);*

fol:axd:ded: derivfacts:c 3. $\{\varphi, \neg\varphi\} \vdash \psi$ *(Ex Falso Quodlibet, Explosion);*

fol:axd:ded: derivfacts:d 4. $\{\neg\neg\varphi\} \vdash \varphi$ *(Double Negation Elimination);*

fol:axd:ded: derivfacts:e 5. *If* $\Gamma \vdash \neg\neg\varphi$ *then* $\Gamma \vdash \varphi$*;*

Problema axd.3. Prove Proposição axd.24

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axd.8 The Deduction Theorem with Quantifiers

fol:axd:ddq: sec

fol:axd:ddq: thm:deduction-thm-q **Teorema axd.25 (Deduction Theorem).** *If* $\Gamma \cup \{\varphi\} \vdash \psi$*, then* $\Gamma \vdash \varphi \to \psi$*.*

Prova. We again proceed by induction on the length of the derivação of ψ from $\Gamma \cup \{\varphi\}$.

The proof of the induction basis is identical to that in the proof of Teorema axd.23.

For the inductive step, suppose again that the derivação of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by an inference rule. If the inference rule is modus ponens, we proceed as in the proof of Teorema axd.23. If the inference rule is QR, we know that $\psi \equiv \chi \to \forall x\, \theta(x)$ and a fórmula of the form $\chi \to \theta(a)$ appears earlier in the derivação, where a does not occur in χ, φ, or Γ. We thus have that

$$\Gamma \cup \{\varphi\} \vdash \chi \to \theta(a),$$

and the induction hypothesis applies, O.s., we have that

$$\Gamma \vdash \varphi \to (\chi \to \theta(a)).$$

By

$$\vdash (\varphi \to (\chi \to \theta(a))) \to ((\varphi \land \chi) \to \theta(a))$$

and modus ponens we get

$$\Gamma \vdash (\varphi \land \chi) \to \theta(a).$$

Since the eigenvariable condition still applies, we can add a step to this derivação justified by QR, and get

$$\Gamma \vdash (\varphi \land \chi) \to \forall x\, \theta(x).$$

We also have

$$\vdash ((\varphi \land \chi) \to \forall x\, \theta(x)) \to (\varphi \to (\chi \to \forall x\, \theta(x)),$$

so by modus ponens,

$$\Gamma \vdash \varphi \to (\chi \to \forall x\, \theta(x)),$$

O.s., $\Gamma \vdash \psi$.

We leave the case where ψ is justified by the rule QR, but is of the form $\exists x\, \theta(x) \to \chi$, as an exercise. □

Problema axd.4. Complete the proof of Teorema axd.25.

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axd.9 Derivabilidade and Consistency

We will now establish a number of properties of the derivabilidade relation. They are independently interesting, but each will play a role in the proof of the completeness theorem. fol:axd:prv:sec

fol:axd:prv:
prop:provability-contr

Proposição axd.26. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Prova. If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \bot$. By Proposição axd.17, $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{\varphi\}$. By Proposição axd.19, $\Gamma \vdash \bot$, O.s., Γ is inconsistent. □

fol:axd:prv:
prop:prov-incons

Proposição axd.27. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Prova. First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by Proposição axd.18. $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by Proposição axd.17. We also have $\vdash \neg\varphi \to (\varphi \to \bot)$ by Eq. (axd.10). So by two applications of Proposição axd.22, we have $\Gamma \cup \{\neg\varphi\} \vdash \bot$.

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, O.s., $\Gamma \cup \{\neg\varphi\} \vdash \bot$. By the deduction theorem, $\Gamma \vdash \neg\varphi \to \bot$. $\Gamma \vdash (\neg\varphi \to \bot) \to \neg\neg\varphi$ by Eq. (axd.13), so $\Gamma \vdash \neg\neg\varphi$ by Proposição axd.22. Since $\Gamma \vdash \neg\neg\varphi \to \varphi$ (Eq. (axd.14)), we have $\Gamma \vdash \varphi$ by Proposição axd.22 again. □

Problema axd.5. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:axd:prv:
prop:explicit-inc

Proposição axd.28. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Prova. $\Gamma \vdash \neg\varphi \to (\varphi \to \bot)$ by Eq. (axd.10). $\Gamma \vdash \bot$ by two applications of Proposição axd.22. □

fol:axd:prv:
prop:provability-exhaustive

Proposição axd.29. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.*

Prova. Exercise. □

Problema axd.6. Prove Proposição axd.29

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axd.10 Derivabilidade and the Propositional Connectives

fol:axd:ppr:
sec

We establish that the derivabilidade relation $\vdash$ of axiomatic deduction is strong enough to establish some basic facts involving the propositional connectives, such as that $\varphi \wedge \psi \vdash \varphi$ and $\varphi, \varphi \to \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem. explanation

fol:axd:ppr:
prop:provability-land

Proposição axd.30.

fol:axd:ppr:
prop:provability-land-left

1. *Both $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$*

fol:axd:ppr:
prop:provability-land-right

2. *$\varphi, \psi \vdash \varphi \wedge \psi$.*

Prova. 1. From Eq. (axd.1) and Eq. (axd.1) by modus ponens.

2. From Eq. (axd.3) by two applications of modus ponens. □

Proposição axd.31. fol:axd:ppr: prop:provability-lor

1. $\varphi \lor \psi, \neg\varphi, \neg\psi$ is inconsistent.

2. Both $\varphi \vdash \varphi \lor \psi$ and $\psi \vdash \varphi \lor \psi$.

Prova. 1. From Eq. (axd.9) we get $\vdash \neg\varphi \to (\varphi \to \bot)$ and $\vdash \neg\psi \to (\psi \to \bot)$. So by the deduction theorem, we have $\{\neg\varphi\} \vdash \varphi \to \bot$ and $\{\neg\psi\} \vdash \psi \to \bot$. From Eq. (axd.6) we get $\{\neg\varphi, \neg\psi\} \vdash (\varphi \lor \psi) \to \bot$. By the deduction theorem, $\{\varphi \lor \psi, \neg\varphi, \neg\psi\} \vdash \bot$.

2. From Eq. (axd.4) and Eq. (axd.5) by modus ponsens. □

Proposição axd.32. fol:axd:ppr: prop:provability-lif

1. $\varphi, \varphi \to \psi \vdash \psi$. fol:axd:ppr: prop:provability-lif-left

2. Both $\neg\varphi \vdash \varphi \to \psi$ and $\psi \vdash \varphi \to \psi$. fol:axd:ppr: prop:provability-lif-right

Prova. 1. We can deriva:

1.	φ	Hyp
2.	$\varphi \to \psi$	Hyp
3.	ψ	1, 2, MP

2. By Eq. (axd.10) and Eq. (axd.7) and the deduction theorem, respectively. □

-NoValue-

axd.11 Derivabilidade and the Quantifiers

explanation The completeness theorem also requires that axiomatic deductions yield the facts about $\vdash$ established in this section. fol:axd:qpr: sec

Teorema axd.33. *If c is a símbolo de constante not occurring in Γ or $\varphi(x)$ and $\Gamma \vdash \varphi(c)$, then $\Gamma \vdash \forall x\, \varphi(x)$.* fol:axd:qpr: thm:strong-generalization

Prova. By the deduction theorem, $\Gamma \vdash \top \to \varphi(c)$. Since c does not occur in Γ or $\top$, we get $\Gamma \vdash \top \to \forall x\, \varphi(x)$. By the deduction theorem again, $\Gamma \vdash \forall x\, \varphi(x)$. □

Proposição axd.34. fol:axd:qpr: prop:provability-quantifiers

1. $\varphi(t) \vdash \exists x\, \varphi(x)$.

2. $\forall x\, \varphi(x) \vdash \varphi(t)$.

Prova. 1. By Eq. (axd.16) and the deduction theorem.

2. By Eq. (axd.15) and the deduction theorem. □

-NoValue-

axd.12 Soundness

fol:axd:sou:sec A derivação system, such as axiomatic deduction, is *sound* if it cannot deriva things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivação systems. Depending on which proof theoretic property is in question, we would like to know for instance, that explanation

1. every derivável φ is valid;
2. if φ is derivável from some others Γ, it is also a consequence of them;
3. if a set of fórmulas Γ is inconsistent, it is unsatisfiable.

These are important properties of a derivação system. If any of them do not hold, the derivação system is deficient—it would deriva too much. Consequently, establishing the soundness of a derivação system is of the utmost importance.

Proposição axd.35. *If φ is an axiom, then $\mathfrak{M}, s \vDash \varphi$ for each estrutura $\mathfrak{M}$ and assignment s.*

Prova. We have to verify that all the axioms are valid. For instance, here is the case for Eq. (axd.15): suppose t is livre para x in φ, and assume $\mathfrak{M}, s \vDash \forall x\, \varphi$. Then by definition of satisfaction, for each $s' \sim_x s$, also $\mathfrak{M}, s' \vDash \varphi$, and in particular this holds when $s'(x) = \mathrm{Val}^{\mathfrak{M}}_s(t)$. By **??**, $\mathfrak{M}, s \vDash \varphi[t/x]$. This shows that $\mathfrak{M}, s \vDash (\forall x\, \varphi \rightarrow \varphi[t/x])$. □

fol:axd:sou:thm:soundness **Teorema axd.36 (Soundness).** *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

Prova. By induction on the length of the derivação of φ from Γ. If there are no steps justified by inferences, then all fórmulas in the derivação are either instances of axioms or are in Γ. By the previous proposition, all the axioms are valid, and hence if φ is an axiom then $\Gamma \vDash \varphi$. If $\varphi \in \Gamma$, then trivially $\Gamma \vDash \varphi$.

If the last step of the derivação of φ is justified by modus ponens, then there are fórmulas ψ and $\psi \rightarrow \varphi$ in the derivação, and the induction hypothesis applies to the part of the derivação ending in those fórmulas (since they contain at least one fewer step justified by an inference). So, by induction hypothesis, $\Gamma \vDash \psi$ and $\Gamma \vDash \psi \rightarrow \varphi$. Then $\Gamma \vDash \varphi$ by **??**.

Now suppose the last step is justified by QR. Then that step has the form $\chi \rightarrow \forall x\, B(x)$ and there is a preceding step $\chi \rightarrow \psi(c)$ with c not in Γ, χ, or $\forall x\, B(x)$. By induction hypothesis, $\Gamma \vDash \chi \rightarrow \psi(c)$. By **??**, $\Gamma \cup \{\chi\} \vDash \psi(c)$.

Consider some structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash \Gamma \cup \{\chi\}$. We need to show that $\mathfrak{M} \vDash \forall x\, \psi(x)$. Since $\forall x\, \psi(x)$ is a sentença, this means we have to show that for every variable assignment s, $\mathfrak{M}, s \vDash \psi(x)$ (**??**). Since $\Gamma \cup \{\chi\}$ consists entirely of sentences, $\mathfrak{M}, s \vDash \theta$ for all $\theta \in \Gamma$ by **??**. Let $\mathfrak{M}'$ be like $\mathfrak{M}$ except that $c^{\mathfrak{M}'} = s(x)$. Since c does not occur in Γ or χ, $\mathfrak{M}' \vDash \Gamma \cup \{\chi\}$ by **??**. Since $\Gamma \cup \{\chi\} \vDash \psi(c)$, $\mathfrak{M}' \vDash B(c)$. Since $\psi(c)$ is a sentença, $\mathfrak{M}', s \vDash \psi(c)$ by **??**. $\mathfrak{M}', s \vDash \psi(x)$ iff $\mathfrak{M}', s \vDash \psi(c)$ by **??** (recall that $\psi(c)$ is just $\psi(x)[c/x]$). So, $\mathfrak{M}', s \vDash \psi(x)$. Since c does not occur in $\psi(x)$, by **??**, $\mathfrak{M}, s \vDash \psi(x)$. But s was an arbitrary variable assignment, so $\mathfrak{M} \vDash \forall x\, \psi(x)$. Thus $\Gamma \cup \{\chi\} \vDash \forall x\, \psi(x)$. By **??**, $\Gamma \vDash \chi \rightarrow \forall x\, \psi(x)$.

The case where φ is justified by QR but is of the form $\exists x\, \psi(x) \rightarrow \chi$ is left as an exercise. □

Problema axd.7. Complete the proof of Teorema axd.36.

Corolário axd.37. *If $\vdash \varphi$, then φ is valid.* fol:axd:sou: cor:weak-soundness

Corolário axd.38. *If Γ is satisfiable, then it is consistent.* fol:axd:sou: cor:consistency-soundness

Prova. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \bot$, O.s., there is a derivação of $\bot$ from Γ. By Teorema axd.36, any estrutura $\mathfrak{M}$ that satisfies Γ must satisfy $\bot$. Since $\mathfrak{M} \nvDash \bot$ for every estrutura $\mathfrak{M}$, no $\mathfrak{M}$ can satisfy Γ, O.s., Γ is not satisfiable. □

-NoValue-

axd.13 Derivações with Predicado de identidade

In order to accommodate $=$ in derivações, we simply add new axiom schemas. The definition of derivação and $\vdash$ remains the same, we just also allow the new axioms. fol:axd:ide: sec

Definição axd.39 (Axioms for predicado de identidade).

$$t = t, \tag{axd.17}$$

fol:axd:ide: ax:id1

$$t_1 = t_2 \rightarrow (\psi(t_1) \rightarrow \psi(t_2)), \tag{axd.18}$$

fol:axd:ide: ax:id2

for any closed terms t, t_1, t_2.

Proposição axd.40. *The axioms Eq. (axd.17) and Eq. (axd.18) are valid.* fol:axd:ide: prop:sound

Prova. Exercise. □

Problema axd.8. Prove Proposição axd.40.

Proposição axd.41. *$\Gamma \vdash t = t$, for any term t and set Γ.* fol:axd:ide: prop:iden1

Proposição axd.42. *If $\Gamma \vdash \varphi(t_1)$ and $\Gamma \vdash t_1 = t_2$, then $\Gamma \vdash \varphi(t_2)$.* fol:axd:ide: prop:iden2

Prova. The fórmula

$$(t_1 = t_2 \rightarrow (\varphi(t_1) \rightarrow \varphi(t_2)))$$

is an instance of Eq. (axd.18). The conclusion follows by two applications of MP. $\square$

Créditos das Imagens

Referências Bibliográficas