und.1 The Decision Problem is Unsolvable

tur:und:uns:sec

tur:und:uns:thm:decision-prob **Theorem und.1.** *The decision problem is unsolvable: There is no Turing machine D, which when started on a tape that contains a sentence ψ of first-order logic as input, D eventually halts, and outputs 1 iff ψ is valid and 0 otherwise.*

Proof. Suppose the decision problem were solvable, i.e., suppose there were a Turing machine D. Then we could solve the halting problem as follows. We construct a Turing machine E that, given as input the number e of Turing machine M_e and input w, computes the corresponding sentence $\tau(M_e, w) \to \alpha(M_e, w)$ and halts, scanning the leftmost square on the tape. The machine $E \frown D$ would then, given input e and w, first compute $\tau(M_e, w) \to \alpha(M_e, w)$ and then run the decision problem machine D on that input. D halts with output 1 iff $\tau(M_e, w) \to \alpha(M_e, w)$ is valid and outputs 0 otherwise. By **??** and **??**, $\tau(M_e, w) \to \alpha(M_e, w)$ is valid iff M_e halts on input w. Thus, $E \frown D$, given input e and w halts with output 1 iff M_e halts on input w and halts with output 0 otherwise. In other words, $E \frown D$ would solve the halting problem. But we know, by **??**, that no such Turing machine can exist. □

tur:und:uns:cor:undecidable-sat **Corollary und.2.** *It is undecidable if an arbitrary sentence of first-order logic is satisfiable.*

Proof. Suppose satisfiability were decidable by a Turing machine S. Then we could solve the decision problem as follows: Given a sentence B as input, move ψ to the right one square. Return to square 1 and write the symbol $\neg$.

Now run the Turing machine S. It eventually halts with output either 1 (if $\neg\psi$ is satisfiable) or 0 (if $\neg\psi$ is unsatisfiable) on the tape. If there is a 1 on square 1, erase it; if square 1 is empty, write a 1, then halt.

This Turing machine always halts, and its output is 1 iff $\neg\psi$ is unsatisfiable and 0 otherwise. Since ψ is valid iff $\neg\psi$ is unsatisfiable, the machine outputs 1 iff ψ is valid, and 0 otherwise, i.e., it would solve the decision problem. □

So there is no Turing machine which always gives a correct "yes" or "no" answer to the question "Is ψ a valid sentence of first-order logic?" However, there *is* a Turing machine that always gives a correct "yes" answer—but simply does not halt if the answer is "no." This follows from the soundness and completeness theorem of first-order logic, and the fact that derivations can be effectively enumerated. explanation

tur:und:uns:thm:valid-ce **Theorem und.3.** *Validity of first-order sentences is semi-decidable: There is a Turing machine E, which when started on a tape that contains a sentence ψ of first-order logic as input, E eventually halts and outputs 1 iff ψ is valid, but does not halt otherwise.*

Proof. All possible derivations of first-order logic can be generated, one after another, by an effective algorithm. The machine E does this, and when it finds a derivation that shows that $\vdash \psi$, it halts with output 1. By the soundness theorem, if E halts with output 1, it's because $\vDash \psi$. By the completeness theorem, if $\vDash \psi$ there is a derivation that shows that $\vdash \psi$. Since E systematically generates all possible derivations, it will eventually find one that shows $\vdash \psi$, so will eventually halt with output 1. □

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Bibliography