siz.1 Cantor's Zig-Zag Method

sfr:siz:zigzag:sec

We've already considered some "easy" enumerations. Now we will consider something a bit harder. Consider the set of pairs of natural numbersdefined by: explanation

$$\mathbb{N} \times \mathbb{N} = \{\langle n, m \rangle : n, m \in \mathbb{N}\}$$

We can organize these ordered pairs into an *array*, like so:

	0	**1**	**2**	**3**	...
0	$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 0,2\rangle$	$\langle 0,3\rangle$	...
1	$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 1,2\rangle$	$\langle 1,3\rangle$	...
2	$\langle 2,0\rangle$	$\langle 2,1\rangle$	$\langle 2,2\rangle$	$\langle 2,3\rangle$	...
3	$\langle 3,0\rangle$	$\langle 3,1\rangle$	$\langle 3,2\rangle$	$\langle 3,3\rangle$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Clearly, every ordered pair in $\mathbb{N} \times \mathbb{N}$ will appear exactly once in the array. In particular, $\langle n, m\rangle$ will appear in the nth row and mth column. But how do we organize the elements of such an array into a "one-dimensional" list? The pattern in the array below demonstrates one way to do this (although of course there are many other options):

	0	**1**	**2**	**3**	**4**	...
0	0	1	3	6	10	...
1	2	4	7	11	...	...
2	5	8	12	...	...	...
3	9	13	...	...	...	...
4	14	...	...	...	...	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\ddots$

This pattern is called *Cantor's zig-zag method*. It enumerates $\mathbb{N} \times \mathbb{N}$ as follows:

$$\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 0,2\rangle, \langle 1,1\rangle, \langle 2,0\rangle, \langle 0,3\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,0\rangle, \ldots$$

And this establishes the following:

sfr:siz:zigzag:natsquaredenumerable

Proposition siz.1. $\mathbb{N} \times \mathbb{N}$ *is enumerable.*

Proof. Let $f\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ take each $k \in \mathbb{N}$ to the tuple $\langle n, m\rangle \in \mathbb{N} \times \mathbb{N}$ such that k is the value of the nth row and mth column in Cantor's zig-zag array. □

This technique also generalises rather nicely. For example, we can use it to enumerate the set of ordered triples of natural numbers, i.e.: explanation

$$\mathbb{N} \times \mathbb{N} \times \mathbb{N} = \{\langle n, m, k\rangle : n, m, k \in \mathbb{N}\}$$

We think of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ as the Cartesian product of $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$, that is,

$$\mathbb{N}^3 = (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} = \{\langle\langle n, m\rangle, k\rangle : n, m, k \in \mathbb{N}\}$$

and thus we can enumerate $\mathbb{N}^3$ with an array by labelling one axis with the enumeration of $\mathbb{N}$, and the other axis with the enumeration of $\mathbb{N}^2$:

	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\dots$
$\langle\mathbf{0},\mathbf{0}\rangle$	$\langle 0,0,0\rangle$	$\langle 0,0,1\rangle$	$\langle 0,0,2\rangle$	$\langle 0,0,3\rangle$	$\dots$
$\langle\mathbf{0},\mathbf{1}\rangle$	$\langle 0,1,0\rangle$	$\langle 0,1,1\rangle$	$\langle 0,1,2\rangle$	$\langle 0,1,3\rangle$	$\dots$
$\langle\mathbf{1},\mathbf{0}\rangle$	$\langle 1,0,0\rangle$	$\langle 1,0,1\rangle$	$\langle 1,0,2\rangle$	$\langle 1,0,3\rangle$	$\dots$
$\langle\mathbf{0},\mathbf{2}\rangle$	$\langle 0,2,0\rangle$	$\langle 0,2,1\rangle$	$\langle 0,2,2\rangle$	$\langle 0,2,3\rangle$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Thus, by using a method like Cantor's zig-zag method, we may similarly obtain an enumeration of $\mathbb{N}^3$. And we can keep going, obtaining enumerations of $\mathbb{N}^n$ for any natural number n. So, we have:

Proposition siz.2. *$\mathbb{N}^n$ is enumerable, for every $n \in \mathbb{N}$.*

Problem siz.1. Show that $(\mathbb{Z}^+)^n$ is enumerable, for every $n \in \mathbb{N}$.

Problem siz.2. Show that $(\mathbb{Z}^+)^*$ is enumerable. You may assume Problem siz.1.

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Bibliography