

siz.1 Cantor's Zig-Zag Method

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We've already considered some "easy" enumerations. Now we will consider something a bit harder. Consider the set of pairs of natural numbers defined by:

explanation

$$\mathbb{N} \times \mathbb{N} = \{\langle n, m \rangle : n, m \in \mathbb{N}\}$$

We can organize these ordered pairs into an *array*, like so:

	0	1	2	3	...
0	$\langle 0, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 2 \rangle$	$\langle 0, 3 \rangle$...
1	$\langle 1, 0 \rangle$	$\langle 1, 1 \rangle$	$\langle 1, 2 \rangle$	$\langle 1, 3 \rangle$...
2	$\langle 2, 0 \rangle$	$\langle 2, 1 \rangle$	$\langle 2, 2 \rangle$	$\langle 2, 3 \rangle$...
3	$\langle 3, 0 \rangle$	$\langle 3, 1 \rangle$	$\langle 3, 2 \rangle$	$\langle 3, 3 \rangle$...
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Clearly, every ordered pair in $\mathbb{N} \times \mathbb{N}$ will appear exactly once in the array. In particular, $\langle n, m \rangle$ will appear in the n th row and m th column. But how do we organize the elements of such an array into a "one-dimensional" list? The pattern in the array below demonstrates one way to do this (although of course there are many other options):

	0	1	2	3	4	...
0	0	1	3	6	10	...
1	2	4	7	11
2	5	8	12
3	9	13
4	14
\vdots	\vdots	\vdots	\vdots	\vdots	...	\ddots

This pattern is called *Cantor's zig-zag method*. It enumerates $\mathbb{N} \times \mathbb{N}$ as follows:

$$\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 1 \rangle, \langle 2, 0 \rangle, \langle 0, 3 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 3, 0 \rangle, \dots$$

And this establishes the following:

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Proposition siz.1. $\mathbb{N} \times \mathbb{N}$ is *enumerable*.

Proof. Let $f: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ take each $k \in \mathbb{N}$ to the tuple $\langle n, m \rangle \in \mathbb{N} \times \mathbb{N}$ such that k is the value of the n th row and m th column in Cantor's zig-zag array. \square

This technique also generalises rather nicely. For example, we can use it to enumerate the set of ordered triples of natural numbers, i.e.:

explanation

$$\mathbb{N} \times \mathbb{N} \times \mathbb{N} = \{\langle n, m, k \rangle : n, m, k \in \mathbb{N}\}$$

We think of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ as the Cartesian product of $\mathbb{N} \times \mathbb{N}$ with \mathbb{N} , that is,

$$\mathbb{N}^3 = (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} = \{\langle \langle n, m \rangle, k \rangle : n, m, k \in \mathbb{N}\}$$

and thus we can enumerate \mathbb{N}^3 with an array by labelling one axis with the enumeration of \mathbb{N} , and the other axis with the enumeration of \mathbb{N}^2 :

	0	1	2	3	...
$\langle 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 1 \rangle$	$\langle 0, 0, 2 \rangle$	$\langle 0, 0, 3 \rangle$...
$\langle 0, 1 \rangle$	$\langle 0, 1, 0 \rangle$	$\langle 0, 1, 1 \rangle$	$\langle 0, 1, 2 \rangle$	$\langle 0, 1, 3 \rangle$...
$\langle 1, 0 \rangle$	$\langle 1, 0, 0 \rangle$	$\langle 1, 0, 1 \rangle$	$\langle 1, 0, 2 \rangle$	$\langle 1, 0, 3 \rangle$...
$\langle 0, 2 \rangle$	$\langle 0, 2, 0 \rangle$	$\langle 0, 2, 1 \rangle$	$\langle 0, 2, 2 \rangle$	$\langle 0, 2, 3 \rangle$...
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Thus, by using a method like Cantor's zig-zag method, we may similarly obtain an enumeration of \mathbb{N}^3 . And we can keep going, obtaining enumerations of \mathbb{N}^n for any natural number n . So, we have:

Proposition siz.2. \mathbb{N}^n is *enumerable*, for every $n \in \mathbb{N}$.

Problem siz.1. Show that $(\mathbb{Z}^+)^n$ is *enumerable*, for every $n \in \mathbb{N}$.

Problem siz.2. Show that $(\mathbb{Z}^+)^*$ is *enumerable*. You may assume **Problem siz.1**.

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Bibliography