siz.1 Pairing Functions and Codes

sfr:siz:pai:sec Cantor's zig-zag method makes the enumerability of $\mathbb{N}^n$ visually evident. But let us focus on our array depicting $\mathbb{N}^2$. Following the zig-zag line in the array and counting the places, we can check that $\langle 1, 2\rangle$ is associated with the number 7. However, it would be nice if we could compute this more directly. That is, it would be nice to have to hand the *inverse* of the zig-zag enumeration, $g\colon \mathbb{N}^2 \to \mathbb{N}$, such that explanation

$$g(\langle 0,0\rangle) = 0,\ g(\langle 0,1\rangle) = 1,\ g(\langle 1,0\rangle) = 2,\ \dots, g(\langle 1,2\rangle) = 7,\ \dots$$

This would enable us to calculate exactly where $\langle n, m\rangle$ will occur in our enumeration.

In fact, we can define g directly by making two observations. First: if the nth row and mth column contains value v, then the $(n+1)$st row and $(m-1)$st column contains value $v+1$. Second: the first row of our enumeration consists of the triangular numbers, starting with 0, 1, 3, 6, etc. The kth triangular number is the sum of the natural numbers $< k$, which can be computed as $k(k+1)/2$. Putting these two observations together, consider this function:

$$g(n, m) = \frac{(n+m+1)(n+m)}{2} + n$$

We often just write $g(n, m)$ rather that $g(\langle n, m\rangle)$, since it is easier on the eyes. This tells you first to determine the $(n+m)^{\text{th}}$ triangle number, and then add n to it. And it populates the array in exactly the way we would like. So in particular, the pair $\langle 1, 2\rangle$ is sent to $\frac{4\times 3}{2} + 1 = 7$.

This function g is the *inverse* of an enumeration of a set of pairs. Such functions are called *pairing functions*.

Definition siz.1 (Pairing function). A function $f\colon A \times B \to \mathbb{N}$ is an arithmetical *pairing function* if f is injective. We also say that f *encodes* $A \times B$, and that $f(x, y)$ is the *code* for $\langle x, y\rangle$.

We can use pairing functions to encode, e.g., pairs of natural numbers; or, in other words, we can represent each *pair* of elements using a *single* number. Using the inverse of the pairing function, we can *decode* the number, i.e., find out which pair it represents. explanation

Problem siz.1. Give an enumeration of the set of all non-negative rational numbers.

Problem siz.2. Show that $\mathbb{Q}$ is enumerable. Recall that any rational number can be written as a fraction z/m with $z \in \mathbb{Z}$, $m \in \mathbb{N}^+$.

Problem siz.3. Define an enumeration of $\mathbb{B}^*$.

Problem siz.4. Recall from your introductory logic course that each possible truth table expresses a truth function. In other words, the truth functions are all functions from $\mathbb{B}^k \to \mathbb{B}$ for some k. Prove that the set of all truth functions is enumerable.

Problem siz.5. Show that the set of all finite subsets of an arbitrary infinite enumerable set is enumerable.

Problem siz.6. A subset of $\mathbb{N}$ is said to be *cofinite* iff it is the complement of a finite set $\mathbb{N}$; that is, $A \subseteq \mathbb{N}$ is cofinite iff $\mathbb{N} \setminus A$ is finite. Let I be the set whose elements are exactly the finite and cofinite subsets of $\mathbb{N}$. Show that I is enumerable.

Problem siz.7. Show that the enumerable union of enumerable sets is enumerable. That is, whenever A_1, A_2, ... are sets, and each A_i is enumerable, then the union $\bigcup_{i=1}^{\infty} A_i$ of all of them is also enumerable. [NB: this is hard!]

Problem siz.8. Let $f\colon A \times B \to \mathbb{N}$ be an arbitrary pairing function. Show that the inverse of f is an enumeration of $A \times B$.

Problem siz.9. Specify a function that encodes $\mathbb{N}^3$.

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Bibliography