siz.1 Non-enumerable Sets

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> This section proves the non-enumerability of $\mathbb{B}^\omega$ and $\wp(\mathbb{N})$ using the definitions in **??**, i.e., requiring a bijection with $\mathbb{N}$ instead of a surjection from $\mathbb{Z}^+$.

The set $\mathbb{N}$ of natural numbers is infinite. It is also trivially enumerable. But the remarkable fact is that there are *non-enumerable* sets, i.e., sets which are not enumerable (see **??**). explanation

This might be surprising. After all, to say that A is non-enumerable is to say that there is *no* bijection $f : \mathbb{N} \to A$; that is, no function mapping the infinitely many elements of $\mathbb{N}$ to A exhausts all of A. So if A is non-enumerable, there are "more" elements of A than there are natural numbers.

To prove that a set is non-enumerable, you have to show that no appropriate bijection can exist. The best way to do this is to show that every attempt to enumerate elements of A must leave at least one element out; this shows that no function $f : \mathbb{N} \to A$ is surjective. And a general strategy for establishing this is to use Cantor's *diagonal method*. Given a list of elements of A, say, x_1, x_2, ..., we construct another element of A which, by its construction, cannot possibly be on that list.

But all of this is best understood by example. So, our first example is the set $\mathbb{B}^\omega$ of all infinite strings of 0's and 1's. (The '$\mathbb{B}$' stands for binary, and we can just think of it as the two-element set $\{0, 1\}$.)

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Theorem siz.1. *$\mathbb{B}^\omega$ is non-enumerable.*

Proof. Consider any enumeration of a subset of $\mathbb{B}^\omega$. So we have some list s_0, s_1, s_2, ... where every s_n is an infinite string of 0's and 1's. Let $s_n(m)$ be the nth digit of the mth string in this list. So we can now think of our list as an array, where $s_n(m)$ is placed at the nth row and mth column:

	0	1	2	3	...
0	$\mathbf{s_0(0)}$	$s_0(1)$	$s_0(2)$	$s_0(3)$	...
1	$s_1(0)$	$\mathbf{s_1(1)}$	$s_1(2)$	$s_1(3)$	...
2	$s_2(0)$	$s_2(1)$	$\mathbf{s_2(2)}$	$s_2(3)$	...
3	$s_3(0)$	$s_3(1)$	$s_3(2)$	$\mathbf{s_3(3)}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We will now construct an infinite string, d, of 0's and 1's which is not on this list. We will do this by specifying each of its entries, i.e., we specify $d(n)$ for all $n \in \mathbb{N}$. Intuitively, we do this by reading down the diagonal of the array above (hence the name "diagonal method") and then changing every 1 to a 0 and every 1 to a 0. More abstractly, we define $d(n)$ to be 0 or 1 according to whether the n-th element of the diagonal, $s_n(n)$, is 1 or 0, that is:

$$d(n) = \begin{cases} 1 & \text{if } s_n(n) = 0 \\ 0 & \text{if } s_n(n) = 1 \end{cases}$$

Clearly $d \in \mathbb{B}^\omega$, since it is an infinite string of 0's and 1's. But we have constructed d so that $d(n) \neq s_n(n)$ for any $n \in \mathbb{N}$. That is, d differs from s_n in its nth entry. So $d \neq s_n$ for any $n \in \mathbb{N}$. So d cannot be on the list s_0, s_1, s_2, …

We have shown, given an arbitrary enumeration of some subset of $\mathbb{B}^\omega$, that it will omit some element of $\mathbb{B}^\omega$. So there is no enumeration of the set $\mathbb{B}^\omega$, i.e., $\mathbb{B}^\omega$ is non-enumerable. □

explanation

This proof method is called "diagonalization" because it uses the diagonal of the array to define d. However, diagonalization need not involve the presence of an array. Indeed, we can show that some set is non-enumerable by using a similar idea, even when no array and no actual diagonal is involved. The following result illustrates how.

Theorem siz.2. *$\wp(\mathbb{N})$ is not enumerable.*

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Proof. We proceed in the same way, by showing that every list of subsets of $\mathbb{N}$ omits some subset of $\mathbb{N}$. So, suppose that we have some list $N_0, N_1, N_2, \dots$ of subsets of $\mathbb{N}$. We define a set D as follows: $n \in D$ iff $n \notin N_n$:

$$D = \{n \in \mathbb{N} : n \notin N_n\}$$

Clearly $D \subseteq \mathbb{N}$. But D cannot be on the list. After all, by construction $n \in D$ iff $n \notin N_n$, so that $D \neq N_n$ for any $n \in \mathbb{N}$. □

explanation

The preceding proof did not mention a diagonal. Still, you can think of it as involving a diagonal if you picture it this way: Imagine the sets N_0, N_1, …, written in an array, where we write N_n on the nth row by writing m in the mth column iff if $m \in N_n$. For example, say the first four sets on that list are $\{0, 1, 2, \dots\}$, $\{1, 3, 5, \dots\}$, $\{0, 1, 4\}$, and $\{2, 3, 4, \dots\}$; then our array would begin with

$$\begin{array}{rllllllll}
N_0 = \{ & \mathbf{0}, & 1, & 2, & & & & \dots\} \\
N_1 = \{ & & \mathbf{1}, & & 3, & & 5, & \dots\} \\
N_2 = \{ & 0, & 1, & & & 4 & & \} \\
N_3 = \{ & & & 2, & \mathbf{3}, & 4, & & \dots\} \\
 & \vdots & & & & & & \ddots
\end{array}$$

Then D is the set obtained by going down the diagonal, placing $n \in D$ iff n is *not* on the diagonal. So in the above case, we would leave out 0 and 1, we would include 2, we would leave out 3, etc.

Problem siz.1. Show that the set of all functions $f\colon \mathbb{N} \to \mathbb{N}$ is non-enumerable by an explicit diagonal argument. That is, show that if f_1, f_2, …, is a list of functions and each $f_i\colon \mathbb{N} \to \mathbb{N}$, then there is some $g\colon \mathbb{N} \to \mathbb{N}$ not on this list.

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Bibliography