

## siz.1 Non-enumerable Sets

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sec

This section proves the non-enumerability of  $\mathbb{B}^\omega$  and  $\wp(\mathbb{N})$  using the definitions in ??, i.e., requiring a bijection with  $\mathbb{N}$  instead of a surjection from  $\mathbb{Z}^+$ .

The set  $\mathbb{N}$  of natural numbers is infinite. It is also trivially **enumerable**. But the remarkable fact is that there are *non-enumerable* sets, i.e., sets which are not **enumerable** (see ??). explanation

This might be surprising. After all, to say that  $A$  is **non-enumerable** is to say that there is *no* **bijection**  $f: \mathbb{N} \rightarrow A$ ; that is, no function mapping the infinitely many **elements** of  $\mathbb{N}$  to  $A$  exhausts all of  $A$ . So if  $A$  is **non-enumerable**, there are “more” **elements** of  $A$  than there are natural numbers.

To prove that a set is **non-enumerable**, you have to show that no appropriate **bijection** can exist. The best way to do this is to show that every attempt to enumerate **elements** of  $A$  must leave at least one **element** out; this shows that no function  $f: \mathbb{N} \rightarrow A$  is **surjective**. And a general strategy for establishing this is to use Cantor’s *diagonal method*. Given a list of **elements** of  $A$ , say,  $x_1, x_2, \dots$ , we construct another **element** of  $A$  which, by its construction, cannot possibly be on that list.

But all of this is best understood by example. So, our first example is the set  $\mathbb{B}^\omega$  of all infinite strings of 0’s and 1’s. (The ‘ $\mathbb{B}$ ’ stands for binary, and we can just think of it as the two-element set  $\{0, 1\}$ .)

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thm:nonenum-bin-omega

**Theorem siz.1.**  $\mathbb{B}^\omega$  is *non-enumerable*.

*Proof.* Consider any enumeration of a subset of  $\mathbb{B}^\omega$ . So we have some list  $s_0, s_1, s_2, \dots$  where every  $s_n$  is an infinite string of 0’s and 1’s. Let  $s_n(m)$  be the  $n$ th digit of the  $m$ th string in this list. So we can now think of our list as an array, where  $s_n(m)$  is placed at the  $n$ th row and  $m$ th column:

	0	1	2	3	...
0	$\mathbf{s_0(0)}$	$s_0(1)$	$s_0(2)$	$s_0(3)$	...
1	$s_1(0)$	$\mathbf{s_1(1)}$	$s_1(2)$	$s_1(3)$	...
2	$s_2(0)$	$s_2(1)$	$\mathbf{s_2(2)}$	$s_2(3)$	...
3	$s_3(0)$	$s_3(1)$	$s_3(2)$	$\mathbf{s_3(3)}$	...
⋮	⋮	⋮	⋮	⋮	⋱

We will now construct an infinite string,  $d$ , of 0’s and 1’s which is not on this list. We will do this by specifying each of its entries, i.e., we specify  $d(n)$  for all  $n \in \mathbb{N}$ . Intuitively, we do this by reading down the diagonal of the array above (hence the name “diagonal method”) and then changing every 1 to a 0

and every 1 to a 0. More abstractly, we define  $d(n)$  to be 0 or 1 according to whether the  $n$ -th **element** of the diagonal,  $s_n(n)$ , is 1 or 0, that is:

$$d(n) = \begin{cases} 1 & \text{if } s_n(n) = 0 \\ 0 & \text{if } s_n(n) = 1 \end{cases}$$

Clearly  $d \in \mathbb{B}^\omega$ , since it is an infinite string of 0's and 1's. But we have constructed  $d$  so that  $d(n) \neq s_n(n)$  for any  $n \in \mathbb{N}$ . That is,  $d$  differs from  $s_n$  in its  $n$ th entry. So  $d \neq s_n$  for any  $n \in \mathbb{N}$ . So  $d$  cannot be on the list  $s_0, s_1, s_2, \dots$

We have shown, given an arbitrary enumeration of some subset of  $\mathbb{B}^\omega$ , that it will omit some **element** of  $\mathbb{B}^\omega$ . So there is no enumeration of the set  $\mathbb{B}^\omega$ , i.e.,  $\mathbb{B}^\omega$  is **non-enumerable**.  $\square$

**explanation**

This proof method is called “diagonalization” because it uses the diagonal of the array to define  $d$ . However, diagonalization need not involve the presence of an array. Indeed, we can show that some set is **non-enumerable** by using a similar idea, even when no array and no actual diagonal is involved. The following result illustrates how.

**Theorem siz.2.**  $\wp(\mathbb{N})$  is not **enumerable**.

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thm:nonenum-pownat*

*Proof.* We proceed in the same way, by showing that every list of subsets of  $\mathbb{N}$  omits some subset of  $\mathbb{N}$ . So, suppose that we have some list  $N_0, N_1, N_2, \dots$  of subsets of  $\mathbb{N}$ . We define a set  $D$  as follows:  $n \in D$  iff  $n \notin N_n$ :

$$D = \{n \in \mathbb{N} : n \notin N_n\}$$

Clearly  $D \subseteq \mathbb{N}$ . But  $D$  cannot be on the list. After all, by construction  $n \in D$  iff  $n \notin N_n$ , so that  $D \neq N_n$  for any  $n \in \mathbb{N}$ .  $\square$

**explanation**

The preceding proof did not mention a diagonal. Still, you can think of it as involving a diagonal if you picture it this way: Imagine the sets  $N_0, N_1, \dots$ , written in an array, where we write  $N_n$  on the  $n$ th row by writing  $m$  in the  $m$ th column iff  $m \in N_n$ . For example, say the first four sets on that list are  $\{0, 1, 2, \dots\}$ ,  $\{1, 3, 5, \dots\}$ ,  $\{0, 1, 4\}$ , and  $\{2, 3, 4, \dots\}$ ; then our array would begin with

$$\begin{array}{ccccccc} N_0 = \{ & \mathbf{0}, & 1, & 2, & & & \dots \} \\ N_1 = \{ & & \mathbf{1}, & & 3, & & 5, & \dots \} \\ N_2 = \{ & 0, & 1, & & & 4 & & \} \\ N_3 = \{ & & & 2, & \mathbf{3}, & 4, & & \dots \} \\ & & & \vdots & & & & \ddots \end{array}$$

Then  $D$  is the set obtained by going down the diagonal, placing  $n \in D$  iff  $n$  is *not* on the diagonal. So in the above case, we would leave out 0 and 1, we would include 2, we would leave out 3, etc.

**Problem siz.1.** Show that the set of all functions  $f: \mathbb{N} \rightarrow \mathbb{N}$  is **non-enumerable** by an explicit diagonal argument. That is, show that if  $f_1, f_2, \dots$ , is a list of functions and each  $f_i: \mathbb{N} \rightarrow \mathbb{N}$ , then there is some  $g: \mathbb{N} \rightarrow \mathbb{N}$  not on this list.

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**Bibliography**