siz.1 Enumerations and Enumerable Sets

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> This section defines enumerations as bijections with (initial segments) of $\mathbb{N}$, the way it's done in set theory. So it conflicts slightly with the definitions in **??**, and repeats all the examples there. It is also a bit more terse than that section.

We can specify finite set is by simply enumerating its elements. We do this when we define a set like so:

$$A = \{a_1, a_2, \dots, a_n\}.$$

Assuming that the elements $a_1, \dots, a_n$ are all distinct, this gives us a bijection between A and the first n natural numbers $0, \dots, n-1$. Conversely, since every finite set has only finitely many elements, every finite set can be put into such a correspondence. In other words, if A is finite, there is a bijection between A and $\{0, \dots, n-1\}$, where n is the number of elements of A.

If we allow for certain kinds of infinite sets, then we will also allow some infinite sets to be enumerated. We can make this precise by saying that an infinite set is enumerated by a bijection between it and all of $\mathbb{N}$.

Definition siz.1 (Enumeration, set-theoretic). An *enumeration* of a set A is a bijection whose range is A and whose domain is either an initial set of natural numbers $\{0, 1, \dots, n\}$ or the entire set of natural numbers $\mathbb{N}$.

There is an intuitive underpinning to this use of the word *enumeration*. For to say that we have enumerated a set A is to say that there is a bijection f which allows us to count out the elements of the set A. The 0th element is $f(0)$, the 1st is $f(1)$, …the nth is $f(n)$….[1] The rationale for this may be made even clearer by adding the following: explanation

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Definition siz.2. A set A is enumerable iff either $A = \emptyset$ or there is an enumeration of A. We say that A is non-enumerable iff A is not enumerable.

So a set is enumerable iff it is empty or you can use an enumeration to count out its elements. explanation

Example siz.3. A function enumerating the natural numbers is simply the identity function $\text{Id}_{\mathbb{N}} \colon \mathbb{N} \to \mathbb{N}$ given by $\text{Id}_{\mathbb{N}}(n) = n$. A function enumerating the *positive* natural numbers, $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$, is the function $g(n) = n + 1$, i.e., the successor function.

[1] Yes, we count from 0. Of course we could also start with 1. This would make no big difference. We would just have to replace $\mathbb{N}$ by $\mathbb{Z}^+$.

Problem siz.1. Show that a set A is enumerable iff either $A = \emptyset$ or there is a surjection $f\colon \mathbb{N} \to A$. Show that A is enumerable iff there is an injection $g\colon A \to \mathbb{N}$.

Example siz.4. The functions $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = 2n \text{ and}$$
$$g(n) = 2n + 1$$

respectively enumerate the even natural numbers and the odd natural numbers. But neither is surjective, so neither is an enumeration of $\mathbb{N}$.

Problem siz.2. Define an enumeration of the square numbers 1, 4, 9, 16, ...

Example siz.5. Let $\lceil x \rceil$ be the *ceiling* function, which rounds x up to the nearest integer. Then the function $f\colon \mathbb{N} \to \mathbb{Z}$ given by:

$$f(n) = (-1)^n \left\lceil \frac{n}{2} \right\rceil$$

enumerates the set of integers $\mathbb{Z}$ as follows:

$$\begin{array}{cccccccc} f(0) & f(1) & f(2) & f(3) & f(4) & f(5) & f(6) & \dots \\ \lceil \frac{0}{2} \rceil & -\lceil \frac{1}{2} \rceil & \lceil \frac{2}{2} \rceil & -\lceil \frac{3}{2} \rceil & \lceil \frac{4}{2} \rceil & -\lceil \frac{5}{2} \rceil & \lceil \frac{6}{2} \rceil & \dots \\ 0 & -1 & 1 & -2 & 2 & -3 & 3 & \dots \end{array}$$

Notice how f generates the values of $\mathbb{Z}$ by "hopping" back and forth between positive and negative integers. You can also think of f as defined by cases as follows:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Problem siz.3. Show that if A and B are enumerable, so is $A \cup B$.

Problem siz.4. Show by induction on n that if A_1, A_2, ..., A_n are all enumerable, so is $A_1 \cup \cdots \cup A_n$.

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Bibliography