siz.1 Sets of Different Sizes, and Cantor's Theorem

sfr:siz:car:sec We have offered a precise statement of the idea that two sets have the same size. We can also offer a precise statement of the idea that one set is smaller than another. Our definition of "is smaller than (or equinumerous)" will require, instead of a bijection between the sets, an injection from the first set to the second. If such a function exists, the size of the first set is less than or equal to the size of the second. Intuitively, an injection from one set to another guarantees that the range of the function has at least as many elements as the domain, since no two elements of the domain map to the same element of the range. explanation

Definition siz.1. A is *no larger than* B, written $A \preceq B$, iff there is an injection $f\colon A \to B$.

It is clear that this is a reflexive and transitive relation, but that it is not symmetric (this is left as an exercise). We can also introduce a notion, which states that one set is (strictly) smaller than another.

Definition siz.2. A is *smaller than* B, written $A \prec B$, iff there is an injection $f\colon A \to B$ but no bijection $g\colon A \to B$, i.e., $A \preceq B$ and $A \not\approx B$.

It is clear that this relation is irreflexive and transitive. (This is left as an exercise.) Using this notation, we can say that a set A is enumerable iff $A \preceq \mathbb{N}$, and that A is non-enumerable iff $\mathbb{N} \prec A$. This allows us to restate **??** as the observation that $\mathbb{Z}^+ \prec \wp(\mathbb{Z}^+)$. In fact, Cantor (1892) proved that this last point is *perfectly general*:

sfr:siz:car:thm:cantor **Theorem siz.3 (Cantor).** *$A \prec \wp(A)$, for any set A.*

Proof. The map $f(x) = \{x\}$ is an injection $f\colon A \to \wp(A)$, since if $x \neq y$, then also $\{x\} \neq \{y\}$ by extensionality, and so $f(x) \neq f(y)$. So we have that $A \preceq \wp(A)$.

> We present the slow proof if **??** is present, otherwise a faster proof matching **??**.

It remains to show that $A \not\approx \wp(A)$. For reductio, suppose $A \approx \wp(A)$, i.e., there is some bijection $g\colon A \to \wp(A)$. Now consider:

$$D = \{x \in A : x \notin g(x)\}$$

Note that $D \subseteq A$, so that $D \in \wp(A)$. Since g is a bijection, there is some $y \in A$ such that $g(y) = D$. But now we have:

$$y \in g(y) \text{ iff } y \in D \text{ iff } y \notin g(y).$$

This is a contradiction; so $A \not\approx \wp(A)$. □

explanation The proof is also worth comparing with the proof of Russell's Paradox, **??**. Indeed, Cantor's Theorem was the inspiration for Russell's own paradox.

Problem siz.1. Show that there cannot be an injection $g\colon \wp(A) \to A$, for any set A. Hint: Suppose $g\colon \wp(A) \to A$ is injective. Consider $D = \{g(B) : B \subseteq A \text{ and } g(B) \notin B\}$. Let $x = g(D)$. Use the fact that g is injective to derive a contradiction.

Photo Credits

Bibliography

Cantor, Georg. 1892. Über eine elementare Frage der Mannigfaltigkeitslehre. *Jahresbericht der deutschen Mathematiker-Vereinigung* 1: 75–8.