

Figure 1: The union  $A \cup B$  of two sets is set of **elements** of  $A$  together with those of  $B$ .

sfr:set:uni:  
fig:union

## set.1 Unions and Intersections

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sec

In ??, we introduced definitions of sets by abstraction, i.e., definitions of the form  $\{x : \varphi(x)\}$ . Here, we invoke some property  $\varphi$ , and this property can mention sets we’ve already defined. So for instance, if  $A$  and  $B$  are sets, the set  $\{x : x \in A \vee x \in B\}$  consists of all those objects which are **elements** of either  $A$  or  $B$ , i.e., it’s the set that combines the **elements** of  $A$  and  $B$ . We can visualize this as in **Figure 1**, where the highlighted area indicates the **elements** of the two sets  $A$  and  $B$  together.

explanation

This operation on sets—combining them—is very useful and common, and so we give it a formal name and a symbol.

**Definition set.1 (Union).** The *union* of two sets  $A$  and  $B$ , written  $A \cup B$ , is the set of all things which are **elements** of  $A$ ,  $B$ , or both.

$$A \cup B = \{x : x \in A \vee x \in B\}$$

**Example set.2.** Since the multiplicity of **elements** doesn’t matter, the union of two sets which have **an element** in common contains that **element** only once, e.g.,  $\{a, b, c\} \cup \{a, 0, 1\} = \{a, b, c, 0, 1\}$ .

The union of a set and one of its subsets is just the bigger set:  $\{a, b, c\} \cup \{a\} = \{a, b, c\}$ .

The union of a set with the empty set is identical to the set:  $\{a, b, c\} \cup \emptyset = \{a, b, c\}$ .

**Problem set.1.** Prove that if  $A \subseteq B$ , then  $A \cup B = B$ .

We can also consider a “dual” operation to union. This is the operation that forms the set of all **elements** that are **elements** of  $A$  and are also **elements** of  $B$ . This operation is called *intersection*, and can be depicted as in **Figure 2**.

explanation

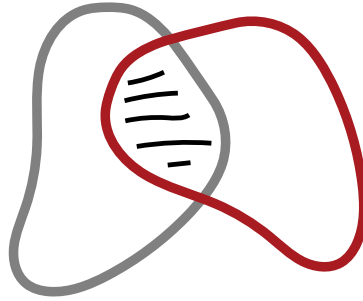


Figure 2: The intersection  $A \cap B$  of two sets is the set of **elements** they have in common.

**Definition set.3 (Intersection).** The *intersection* of two sets  $A$  and  $B$ , written  $A \cap B$ , is the set of all things which are **elements** of both  $A$  and  $B$ .

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fig:intersection

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

Two sets are called *disjoint* if their intersection is empty. This means they have no **elements** in common.

**Example set.4.** If two sets have no **elements** in common, their intersection is empty:  $\{a, b, c\} \cap \{0, 1\} = \emptyset$ .

If two sets do have **elements** in common, their intersection is the set of all those:  $\{a, b, c\} \cap \{a, b, d\} = \{a, b\}$ .

The intersection of a set with one of its subsets is just the smaller set:  $\{a, b, c\} \cap \{a, b\} = \{a, b\}$ .

The intersection of any set with the empty set is empty:  $\{a, b, c\} \cap \emptyset = \emptyset$ .

**Problem set.2.** Prove rigorously that if  $A \subseteq B$ , then  $A \cap B = A$ .

explanation

We can also form the union or intersection of more than two sets. An elegant way of dealing with this in general is the following: suppose you collect all the sets you want to form the union (or intersection) of into a single set. Then we can define the union of all our original sets as the set of all objects which belong to at least one **element** of the set, and the intersection as the set of all objects which belong to every **element** of the set.

**Definition set.5.** If  $A$  is a set of sets, then  $\bigcup A$  is the set of **elements** of **elements** of  $A$ :

$$\begin{aligned} \bigcup A &= \{x : x \text{ belongs to an element of } A\}, \text{ i.e.,} \\ &= \{x : \text{there is a } B \in A \text{ so that } x \in B\} \end{aligned}$$

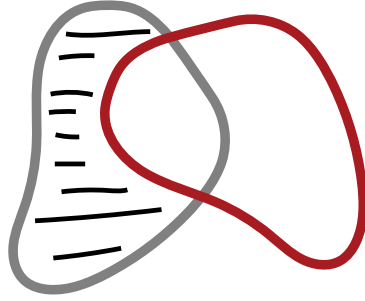


Figure 3: The difference  $A \setminus B$  of two sets is the set of those **elements** of  $A$  which are not also **elements** of  $B$ .

sfr:set:uni:  
difference

**Definition set.6.** If  $A$  is a set of sets, then  $\bigcap A$  is the set of objects which all elements of  $A$  have in common:

$$\begin{aligned}\bigcap A &= \{x : x \text{ belongs to every element of } A\}, \text{ i.e.,} \\ &= \{x : \text{for all } B \in A, x \in B\}\end{aligned}$$

**Example set.7.** Suppose  $A = \{\{a, b\}, \{a, d, e\}, \{a, d\}\}$ . Then  $\bigcup A = \{a, b, d, e\}$  and  $\bigcap A = \{a\}$ .

**Problem set.3.** Show that if  $A$  is a set and  $A \in B$ , then  $A \subseteq \bigcup B$ .

We could also do the same for a sequence of sets  $A_1, A_2, \dots$

$$\begin{aligned}\bigcup_i A_i &= \{x : x \text{ belongs to one of the } A_i\} \\ \bigcap_i A_i &= \{x : x \text{ belongs to every } A_i\}.\end{aligned}$$

When we have an *index* of sets, i.e., some set  $I$  such that we are considering  $A_i$  for each  $i \in I$ , we may also use these abbreviations:

$$\begin{aligned}\bigcup_{i \in I} A_i &= \bigcup \{A_i : i \in I\} \\ \bigcap_{i \in I} A_i &= \bigcap \{A_i : i \in I\}\end{aligned}$$

Finally, we may want to think about the set of all **elements** in  $A$  which are not in  $B$ . We can depict this as in **Figure 3**.

**Definition set.8 (Difference).** The *set difference*  $A \setminus B$  is the set of all **elements** of  $A$  which are not also **elements** of  $B$ , i.e.,

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

**Problem set.4.** Prove that if  $A \subsetneq B$ , then  $B \setminus A \neq \emptyset$ .

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**Bibliography**