Special Properties of Relations

Some kinds of relations turn out to be so common that they have been given special names. For instance, $\leq$ and $\subseteq$ both relate their respective domains (say, $\mathbb{N}$ in the case of $\leq$ and $\mathcal{P}(A)$ in the case of $\subseteq$) in similar ways. To get at exactly how these relations are similar, and how they differ, we categorize them according to some special properties that relations can have. It turns out that (combinations of) some of these special properties are especially important: orders and equivalence relations.

**Definition rel.1 (Reflexivity).** A relation $R \subseteq A^2$ is reflexive iff, for every $x \in A$, $Rx x$.

**Definition rel.2 (Transitivity).** A relation $R \subseteq A^2$ is transitive iff, whenever $Rxy$ and $Ryz$, then also $Rxz$.

**Definition rel.3 (Symmetry).** A relation $R \subseteq A^2$ is symmetric iff, whenever $Rxy$, then also $Ryx$.

**Definition rel.4 (Anti-symmetry).** A relation $R \subseteq A^2$ is anti-symmetric iff, whenever both $Rxy$ and $Ryx$, then $x = y$ (or, in other words: if $x \neq y$ then either $\neg Rxy$ or $\neg Ryx$).

In a symmetric relation, $Rxy$ and $Ryx$ always hold together, or neither holds. In an anti-symmetric relation, the only way for $Rxy$ and $Ryx$ to hold together is if $x = y$. Note that this does not require that $Rxy$ and $Ryx$ holds when $x = y$, only that it isn’t ruled out. So an anti-symmetric relation can be reflexive, but it is not the case that every anti-symmetric relation is reflexive. Also note that being anti-symmetric and merely not being symmetric are different conditions. In fact, a relation can be both symmetric and anti-symmetric at the same time (e.g., the identity relation is).

**Definition rel.5 (Connectivity).** A relation $R \subseteq A^2$ is connected if for all $x, y \in A$, if $x \neq y$, then either $Rxy$ or $Ryx$.

**Problem rel.1.** Give examples of relations that are (a) reflexive and symmetric but not transitive, (b) reflexive and anti-symmetric, (c) anti-symmetric, transitive, but not reflexive, and (d) reflexive, symmetric, and transitive. Do not use relations on numbers or sets.

**Definition rel.6 (Irreflexivity).** A relation $R \subseteq A^2$ is called irreflexive if, for all $x \in A$, not $Rx x$.

**Definition rel.7 (Asymmetry).** A relation $R \subseteq A^2$ is called asymmetric if for no pair $x, y \in A$ we have both $Rxy$ and $Ryx$. 

1
Note that if \( A \neq \emptyset \), then no irreflexive relation on \( A \) is reflexive and every asymmetric relation on \( A \) is also anti-symmetric. However, there are \( R \subseteq A^2 \) that are not reflexive and also not irreflexive, and there are anti-symmetric relations that are not asymmetric.

**Photo Credits**

**Bibliography**