

rel.1 Orders

sfr:rel:ord:sec Many of our comparisons involve describing some objects as being “less than”, explanation “equal to”, or “greater than” other objects, in a certain respect. These involve *order* relations. But there are different kinds of order relations. For instance, some require that any two objects be comparable, others don’t. Some include identity (like \leq) and some exclude it (like $<$). It will help us to have a taxonomy here.

Definition rel.1 (Preorder). A relation which is both reflexive and transitive is called a *preorder*.

Definition rel.2 (Partial order). A preorder which is also anti-symmetric is called a *partial order*.

sfr:rel:ord:def:linearorder **Definition rel.3 (Linear order).** A partial order which is also connected is called a *total order* or *linear order*.

Example rel.4. Every linear order is also a partial order, and every partial order is also a preorder, but the converses don’t hold. The universal relation on A is a preorder, since it is reflexive and transitive. But, if A has more than one element, the universal relation is not anti-symmetric, and so not a partial order.

Example rel.5. Consider the *no longer than* relation \preceq on \mathbb{B}^* : $x \preceq y$ iff $\text{len}(x) \leq \text{len}(y)$. This is a preorder (reflexive and transitive), and even connected, but not a partial order, since it is not anti-symmetric. For instance, $01 \preceq 10$ and $10 \preceq 01$, but $01 \neq 10$.

Example rel.6. An important partial order is the relation \subseteq on a set of sets. This is not in general a linear order, since if $a \neq b$ and we consider $\wp(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, we see that $\{a\} \not\subseteq \{b\}$ and $\{a\} \neq \{b\}$ and $\{b\} \not\subseteq \{a\}$.

Example rel.7. The relation of *divisibility without remainder* gives us a partial order which isn’t a linear order. For integers n, m , we write $n \mid m$ to mean n (evenly) divides m , i.e., iff there is some integer k so that $m = kn$. On \mathbb{N} , this is a partial order, but not a linear order: for instance, $2 \nmid 3$ and also $3 \nmid 2$. Considered as a relation on \mathbb{Z} , divisibility is only a preorder since it is not anti-symmetric: $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$.

Definition rel.8 (Strict order). A *strict order* is a relation which is irreflexive, asymmetric, and transitive.

sfr:rel:ord:def:strictlinearorder **Definition rel.9 (Strict linear order).** A strict order which is also connected is called a *strict total order* or *strict linear order*.

Example rel.10. \leq is the linear order corresponding to the strict linear order $<$. \subseteq is the partial order corresponding to the strict order \subsetneq .

Any strict order R on A can be turned into a partial order by adding the diagonal Id_A , i.e., adding all the pairs $\langle x, x \rangle$. (This is called the *reflexive closure* of R .) Conversely, starting from a partial order, one can get a strict order by removing Id_A . These next two results make this precise.

Proposition rel.11. *If R is a strict order on A , then $R^+ = R \cup \text{Id}_A$ is a partial order. Moreover, if R is a strict linear order, then R^+ is a linear order.* [sfr:rel:ord:](#)
[prop:stricttopartial](#)

Proof. Suppose R is a strict order, i.e., $R \subseteq A^2$ and R is irreflexive, asymmetric, and transitive. Let $R^+ = R \cup \text{Id}_A$. We have to show that R^+ is reflexive, anti-symmetric, and transitive.

R^+ is clearly reflexive, since $\langle x, x \rangle \in \text{Id}_A \subseteq R^+$ for all $x \in A$.

To show R^+ is anti-symmetric, suppose for reductio that R^+xy and R^+yx but $x \neq y$. Since $\langle x, y \rangle \in R \cup \text{Id}_A$, but $\langle x, y \rangle \notin \text{Id}_A$, we must have $\langle x, y \rangle \in R$, i.e., Rxy . Similarly, Ryx . But this contradicts the assumption that R is asymmetric.

To establish transitivity, suppose that R^+xy and R^+yz . If both $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$, then $\langle x, z \rangle \in R$ since R is transitive. Otherwise, either $\langle x, y \rangle \in \text{Id}_A$, i.e., $x = y$, or $\langle y, z \rangle \in \text{Id}_A$, i.e., $y = z$. In the first case, we have that R^+yz by assumption, $x = y$, hence R^+xz . Similarly in the second case. In either case, R^+xz , thus, R^+ is also transitive.

Concerning the “moreover” clause, suppose that R is also connected. So for all $x \neq y$, either Rxy or Ryx , i.e., either $\langle x, y \rangle \in R$ or $\langle y, x \rangle \in R$. Since $R \subseteq R^+$, this remains true of R^+ , so R^+ is connected as well. \square

Proposition rel.12. *If R is a partial order on A , then $R^- = R \setminus \text{Id}_A$ is a strict order. Moreover, if R is a linear order, then R^- is a strict linear order.* [sfr:rel:ord:](#)
[prop:partialtostrict](#)

Proof. This is left as an exercise. \square

Problem rel.1. Give a proof of [Proposition rel.12](#).

The following simple result establishes that strict linear orders satisfy an extensionality-like property:

Proposition rel.13. *If $<$ is a strict linear order on A , then:* [sfr:rel:ord:](#)
[prop:extensionality-strictlinearord](#)

$$(\forall a, b \in A)((\forall x \in A)(x < a \leftrightarrow x < b) \rightarrow a = b).$$

Proof. Suppose $(\forall x \in A)(x < a \leftrightarrow x < b)$. If $a < b$, then $a < a$, contradicting the fact that $<$ is irreflexive; so $a \not< b$. Exactly similarly, $b \not< a$. So $a = b$, as $<$ is connected. \square

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Bibliography