rel.1 Operations on Relations

sfr:rel:ops:sec It is often useful to modify or combine relations. In **??**, we considered the *union* of relations, which is just the union of two relations considered as sets of pairs. Similarly, in **??**, we considered the relative difference of relations. Here are some other operations we can perform on relations.

sfr:rel:ops:relationoperations **Definition rel.1.** Let R, S be relations, and A be any set.

The *inverse* of R is $R^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in R\}$.

The *relative product* of R and S is $(R \mid S) = \{\langle x, z\rangle : \exists y(Rxy \land Syz)\}$.

The *restriction* of R to A is $R\restriction_A = R \cap A^2$.

The *application* of R to A is $R[A] = \{y : (\exists x \in A)Rxy\}$

Example rel.2. Let $S \subseteq \mathbb{Z}^2$ be the successor relation on $\mathbb{Z}$, i.e., $S = \{\langle x, y\rangle \in \mathbb{Z}^2 : x + 1 = y\}$, so that Sxy iff $x + 1 = y$.

S^{-1} is the predecessor relation on $\mathbb{Z}$, i.e., $\{\langle x, y\rangle \in \mathbb{Z}^2 : x - 1 = y\}$.

$S \mid S$ is $\{\langle x, y\rangle \in \mathbb{Z}^2 : x + 2 = y\}$

$S\restriction_{\mathbb{N}}$ is the successor relation on $\mathbb{N}$.

$S[\{1, 2, 3\}]$ is $\{2, 3, 4\}$.

Definition rel.3 (Transitive closure). Let $R \subseteq A^2$ be a binary relation.

The *transitive closure* of R is $R^+ = \bigcup_{0<n\in\mathbb{N}} R^n$, where we recursively define $R^1 = R$ and $R^{n+1} = R^n \mid R$.

The *reflexive transitive closure* of R is $R^* = R^+ \cup \mathrm{Id}_A$.

Example rel.4. Take the successor relation $S \subseteq \mathbb{Z}^2$. S^2xy iff $x + 2 = y$, S^3xy iff $x + 3 = y$, etc. So S^+xy iff $x + n = y$ for some $n \geq 1$. In other words, S^+xy iff $x < y$, and S^*xy iff $x \leq y$.

Problem rel.1. Show that the transitive closure of R is in fact transitive.

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Bibliography