rel.1 Equivalence Relations

The identity relation on a set is reflexive, symmetric, and transitive. Relations $R$ that have all three of these properties are very common.

**Definition rel.1 (Equivalence relation).** A relation $R \subseteq A^2$ that is reflexive, symmetric, and transitive is called an *equivalence relation*. Elements $x$ and $y$ of $A$ are said to be $R$-equivalent if $R_{xy}$.

Equivalence relations give rise to the notion of an *equivalence class*. An equivalence relation “chunks up” the domain into different partitions. Within each partition, all the objects are related to one another; and no objects from different partitions relate to one another. Sometimes, it’s helpful just to talk about these partitions directly. To that end, we introduce a definition:

**Definition rel.2.** Let $R \subseteq A^2$ be an equivalence relation. For each $x \in A$, the *equivalence class* of $x$ in $A$ is the set $[x]_R = \{y \in A : R_{xy}\}$. The quotient of $A$ under $R$ is $A/R = \{[x]_R : x \in A\}$, i.e., the set of these equivalence classes.

The next result vindicates the definition of an equivalence class, in proving that the equivalence classes are indeed the partitions of $A$:

**Proposition rel.3.** If $R \subseteq A^2$ is an equivalence relation, then $R_{xy}$ iff $[x]_R = [y]_R$.

*Proof.* For the left-to-right direction, suppose $R_{xy}$, and let $z \in [x]_R$. By definition, then, $R_{xz}$. Since $R$ is an equivalence relation, $R_{yz}$. (Spelling this out: as $R_{xy}$ and $R$ is symmetric we have $R_{yx}$, and as $R_{xz}$ and $R$ is transitive we have $R_{yz}$.) So $z \in [y]_R$. Generalising, $[x]_R \subseteq [y]_R$. But exactly similarly, $[y]_R \subseteq [x]_R$. So $[x]_R = [y]_R$, by extensionality.

For the right-to-left direction, suppose $[x]_R = [y]_R$. Since $R$ is reflexive, $R_{yy}$, so $y \in [y]_R$. Thus also $y \in [x]_R$ by the assumption that $[x]_R = [y]_R$. So $R_{xy}$. □

**Example rel.4.** A nice example of equivalence relations comes from modular arithmetic. For any $a$, $b$, and $n \in \mathbb{Z}^+$, say that $a \equiv_n b$ iff dividing $a$ by $n$ gives the same remainder as dividing $b$ by $n$. (Somewhat more symbolically: $a \equiv_n b$ iff, for some $k \in \mathbb{Z}$, $a - b = kn$.) Now, $\equiv_n$ is an equivalence relation, for any $n$. And there are exactly $n$ distinct equivalence classes generated by $\equiv_n$; that is, $\mathbb{N}/\equiv_n$ has $n$ elements. These are: the set of numbers divisible by $n$ without remainder, i.e., $[0]_\equiv_n$; the set of numbers divisible by $n$ with remainder 1, i.e., $[1]_\equiv_n$; . . . ; and the set of numbers divisible by $n$ with remainder $n - 1$, i.e., $[n - 1]_\equiv_n$.

**Problem rel.1.** Show that $\equiv_n$ is an equivalence relation, for any $n \in \mathbb{Z}^+$, and that $\mathbb{N}/\equiv_n$ has exactly $n$ members.
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Bibliography