infinite.1 Hilbert's Hotel

sfr:infinite:hilbert:sec

The set of the natural numbers is obviously infinite. So, if we do not want to *help ourselves* to the natural numbers, our first step must be characterize an infinite set in terms that do not require mentioning the natural numbers themselves. Here is a nice approach, presented by Hilbert in a lecture from 1924. He asks us to imagine

> [...] a hotel with a finite number of rooms. All of these rooms should be occupied by exactly one guest. If the guests now swap their rooms somehow, [but] so that each room still contains no more than one person, then no rooms will become free, and the hotel-owner cannot in this way create a new place for a newly arriving guest [...¶...]
>
> Now we stipulate that the hotel shall have infinitely many numbered rooms 1, 2, 3, 4, 5, ..., each of which is occupied by exactly one guest. As soon as a new guest comes along, the owner only needs to move each of the old guests into the room associated with the number one higher, and room 1 will be free for the newly-arriving guest.
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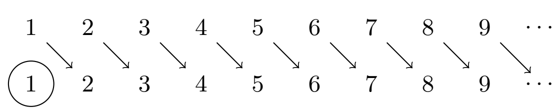
> 
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> (published in Hilbert 2013, 730; our translation)

The crucial point is that Hilbert's Hotel has infinitely many rooms; and we can take his explanation to define what it means to say this. Indeed, this was Dedekind's approach (presented here, of course, with massive anachronism; Dedekind's definition is from 1888):

sfr:infinite:hilbert:defn:DedekindInfinite

Definition infinite.1. A set A is *Dedekind infinite* iff there is an injection from A to a proper subset of A. That is, there is some $o \in A$ and an injection $f \colon A \to A$ such that $o \notin \operatorname{ran}(f)$.

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Bibliography

Dedekind, Richard. 1888. *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg.

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