

## fun.1 Functions as Relations

sfr:fun:rel: sec A function which maps **elements** of  $A$  to **elements** of  $B$  obviously defines a explanation relation between  $A$  and  $B$ , namely the relation which holds between  $x$  and  $y$  iff  $f(x) = y$ . In fact, we might even—if we are interested in reducing the building blocks of mathematics for instance—*identify* the function  $f$  with this relation, i.e., with a set of pairs. This then raises the question: which relations define functions in this way?

**Definition fun.1 (Graph of a function).** Let  $f: A \rightarrow B$  be a function. The *graph* of  $f$  is the relation  $R_f \subseteq A \times B$  defined by

$$R_f = \{\langle x, y \rangle : f(x) = y\}.$$

The graph of a function is uniquely determined, by extensionality. explanation Moreover, extensionality (on sets) will immediately vindicate the implicit principle of extensionality for functions, whereby if  $f$  and  $g$  share a domain and codomain then they are identical if they agree on all values.

Similarly, if a relation is “functional”, then it is the graph of a function.

sfr:fun:rel: prop:graph-function **Proposition fun.2.** Let  $R \subseteq A \times B$  be such that:

1. If  $Rxy$  and  $Rxz$  then  $y = z$ ; and
2. for every  $x \in A$  there is some  $y \in B$  such that  $\langle x, y \rangle \in R$ .

Then  $R$  is the graph of the function  $f: A \rightarrow B$  defined by  $f(x) = y$  iff  $Rxy$ .

*Proof.* Suppose there is a  $y$  such that  $Rxy$ . If there were another  $z \neq y$  such that  $Rxz$ , the condition on  $R$  would be violated. Hence, if there is a  $y$  such that  $Rxy$ , this  $y$  is unique, and so  $f$  is well-defined. Obviously,  $R_f = R$ .  $\square$

Every function  $f: A \rightarrow B$  has a graph, i.e., a relation on  $A \times B$  defined by  $f(x) = y$ . explanation On the other hand, every relation  $R \subseteq A \times B$  with the properties given in **Proposition fun.2** is the graph of a function  $f: A \rightarrow B$ . Because of this close connection between functions and their graphs, we can think of a function simply as its graph. In other words, functions can be identified with certain relations, i.e., with certain sets of tuples. We can now consider performing similar operations on functions as we performed on relations (see ??). In particular:

sfr:fun:rel: defn:funimage **Definition fun.3.** Let  $f: A \rightarrow B$  be a function with  $C \subseteq A$ .

The *restriction* of  $f$  to  $C$  is the function  $f|_C: C \rightarrow B$  defined by  $(f|_C)(x) = f(x)$  for all  $x \in C$ . In other words,  $f|_C = \{\langle x, y \rangle \in R_f : x \in C\}$ .

The *application* of  $f$  to  $C$  is  $f[C] = \{f(x) : x \in C\}$ . We also call this the *image* of  $C$  under  $f$ .

**explanation** It follows from these definitions that  $\text{ran}(f) = f[\text{dom}(f)]$ , for any function  $f$ .

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## Bibliography