

Figure 1: A surjective function has every element of the codomain as a value.

sfr:fun:kin:  
fig:surjective



Figure 2: An injective function never maps two different arguments to the same value.

sfr:fun:kin:  
fig:injective

## fun.1 Kinds of Functions

It will be useful to introduce a kind of taxonomy for some of the kinds of functions which we encounter most frequently. explanation

To start, we might want to consider functions which have the property that every member of the codomain is a value of the function. Such functions are called **surjective**, and can be pictured as in **Figure 1**.

**Definition fun.1 (Surjective function).** A function  $f: A \rightarrow B$  is *surjective* iff  $B$  is also the range of  $f$ , i.e., for every  $y \in B$  there is at least one  $x \in A$  such that  $f(x) = y$ , or in symbols:

$$(\forall y \in B)(\exists x \in A)f(x) = y.$$

We call such a function a **surjection** from  $A$  to  $B$ .

If you want to show that  $f$  is a **surjection**, then you need to show that every object in  $f$ 's codomain is the value of  $f(x)$  for some input  $x$ . explanation

Note that any function *induces a surjection*. After all, given a function  $f: A \rightarrow B$ , let  $f': A \rightarrow \text{ran}(f)$  be defined by  $f'(x) = f(x)$ . Since  $\text{ran}(f)$  is defined as  $\{f(x) \in B : x \in A\}$ , this function  $f'$  is guaranteed to be a **surjection**.

Now, any function maps each possible input to a unique output. But there are also functions which never map different inputs to the same outputs. Such functions are called **injective**, and can be pictured as in **Figure 2**. explanation

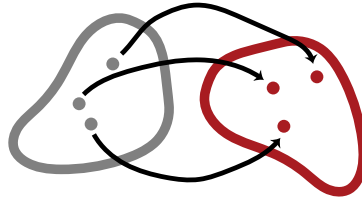


Figure 3: A **bijective** function uniquely pairs the elements of the codomain with those of the domain.

**Definition fun.2 (Injective function).** A function  $f: A \rightarrow B$  is *injective* iff for each  $y \in B$  there is at most one  $x \in A$  such that  $f(x) = y$ . We call such a function an **injection** from  $A$  to  $B$ .

sfr:fun:kin:  
fig:bijective

**explanation** If you want to show that  $f$  is an **injection**, you need to show that for any **elements**  $x$  and  $y$  of  $f$ 's domain, if  $f(x) = f(y)$ , then  $x = y$ .

**Example fun.3.** The constant function  $f: \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = 1$  is neither **injective**, nor **surjective**.

The identity function  $f: \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x$  is both **injective** and **surjective**.

The successor function  $f: \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x + 1$  is **injective** but not **surjective**.

The function  $f: \mathbb{N} \rightarrow \mathbb{N}$  defined by:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

is **surjective**, but not **injective**.

**explanation** Often enough, we want to consider functions which are both **injective** and **surjective**. We call such functions **bijective**. They look like the function pictured in **Figure 3**. **Bijections** are also sometimes called *one-to-one correspondences*, since they uniquely pair elements of the codomain with elements of the domain.

**Definition fun.4 (Bijection).** A function  $f: A \rightarrow B$  is *bijective* iff it is both **surjective** and **injective**. We call such a function a **bijection** from  $A$  to  $B$  (or between  $A$  and  $B$ ).

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## Bibliography