



Figure 1: A function is a mapping of each **element** of one set to **an element** of another. An arrow points from an argument in the domain to the corresponding value in the codomain.

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fig:function

## fun.1 Basics

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A *function* is a map which sends each **element** of a given set to a specific **element** in some (other) given set. For instance, the operation of adding 1 defines a function: each number  $n$  is mapped to a unique number  $n + 1$ .

explanation

More generally, functions may take pairs, triples, etc., as inputs and returns some kind of output. Many functions are familiar to us from basic arithmetic. For instance, addition and multiplication are functions. They take in two numbers and return a third.

In this mathematical, abstract sense, a function is a *black box*: what matters is only what output is paired with what input, not the method for calculating the output.

**Definition fun.1 (Function).** A *function*  $f: A \rightarrow B$  is a mapping of each **element** of  $A$  to an **element** of  $B$ .

We call  $A$  the *domain* of  $f$  and  $B$  the *codomain* of  $f$ . The **elements** of  $A$  are called inputs or *arguments* of  $f$ , and the **element** of  $B$  that is paired with an argument  $x$  by  $f$  is called the *value of  $f$*  for argument  $x$ , written  $f(x)$ .

The *range*  $\text{ran}(f)$  of  $f$  is the subset of the codomain consisting of the values of  $f$  for some argument;  $\text{ran}(f) = \{f(x) : x \in A\}$ .

The diagram in **Figure 1** may help to think about functions. The ellipse on the left represents the function's *domain*; the ellipse on the right represents the function's *codomain*; and an arrow points from an *argument* in the domain to the corresponding *value* in the codomain.

**Example fun.2.** Multiplication takes pairs of natural numbers as inputs and maps them to natural numbers as outputs, so goes from  $\mathbb{N} \times \mathbb{N}$  (the domain) to  $\mathbb{N}$  (the codomain). As it turns out, the range is also  $\mathbb{N}$ , since every  $n \in \mathbb{N}$  is  $n \times 1$ .

**Example fun.3.** Multiplication is a function because it pairs each input—each pair of natural numbers—with a single output:  $\times: \mathbb{N}^2 \rightarrow \mathbb{N}$ . By contrast, the square root operation applied to the domain  $\mathbb{N}$  is not functional, since each positive integer  $n$  has two square roots:  $\sqrt{n}$  and  $-\sqrt{n}$ . We can make it functional by only returning the positive square root:  $\sqrt{\phantom{x}}: \mathbb{N} \rightarrow \mathbb{R}$ .

**Example fun.4.** The relation that pairs each student in a class with their final grade is a function—no student can get two different final grades in the same class. The relation that pairs each student in a class with their parents is not a function: students can have zero, or two, or more parents.

explanation

We can define functions by specifying in some precise way what the value of the function is for every possible argument. Different ways of doing this are by giving a formula, describing a method for computing the value, or listing the values for each argument. However functions are defined, we must make sure that for each argument we specify one, and only one, value.

**Example fun.5.** Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be defined such that  $f(x) = x + 1$ . This is a definition that specifies  $f$  as a function which takes in natural numbers and outputs natural numbers. It tells us that, given a natural number  $x$ ,  $f$  will output its successor  $x + 1$ . In this case, the codomain  $\mathbb{N}$  is not the range of  $f$ , since the natural number 0 is not the successor of any natural number. The range of  $f$  is the set of all positive integers,  $\mathbb{Z}^+$ .

**Example fun.6.** Let  $g: \mathbb{N} \rightarrow \mathbb{N}$  be defined such that  $g(x) = x + 2 - 1$ . This tells us that  $g$  is a function which takes in natural numbers and outputs natural numbers. Given a natural number  $n$ ,  $g$  will output the predecessor of the successor of the successor of  $x$ , i.e.,  $x + 1$ .

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explanation

We just considered two functions,  $f$  and  $g$ , with different *definitions*. However, these are the *same function*. After all, for any natural number  $n$ , we have that  $f(n) = n + 1 = n + 2 - 1 = g(n)$ . Otherwise put: our definitions for  $f$  and  $g$  specify the same mapping by means of different equations. Implicitly, then, we are relying upon a principle of extensionality for functions,

$$\text{if } \forall x f(x) = g(x), \text{ then } f = g$$

provided that  $f$  and  $g$  share the same domain and codomain.

**Example fun.7.** We can also define functions by cases. For instance, we could define  $h: \mathbb{N} \rightarrow \mathbb{N}$  by

$$h(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

Since every natural number is either even or odd, the output of this function will always be a natural number. Just remember that if you define a function by cases, every possible input must fall into exactly one case. In some cases, this will require a proof that the cases are exhaustive and exclusive.

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## Bibliography