fun.1 Composition of Functions

We can define a new function by composing two functions, $f$ and $g$, i.e., by first applying $f$ and then $g$. Of course, this is only possible if the ranges and domains match, i.e., the range of $f$ must be a subset of the domain of $g$.

A diagram might help to explain the idea of composition. In Figure 1, we depict two functions $f: A \rightarrow B$ and $g: B \rightarrow C$ and their composition $(g \circ f)$. The function $(g \circ f): A \rightarrow C$ pairs each element of $A$ with an element of $C$.

We specify which element of $C$ an element of $A$ is paired with as follows: given an input $x \in A$, first apply the function $f$ to $x$, which will output some $f(x) = y \in B$, then apply the function $g$ to $y$, which will output some $g(f(x)) = g(y) = z \in C$.

**Definition fun.1 (Composition).** Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions. The composition of $f$ with $g$ is $g \circ f: A \rightarrow C$, where $(g \circ f)(x) = g(f(x))$.

**Example fun.2.** Consider the functions $f(x) = x + 1$, and $g(x) = 2x$. Since $(g \circ f)(x) = g(f(x))$, for each input $x$ you must first take its successor, then multiply the result by two. So their composition is given by $(g \circ f)(x) = 2(x+1)$.

**Problem fun.1.** Show that if $f: A \rightarrow B$ and $g: B \rightarrow C$ are both injective, then $g \circ f: A \rightarrow C$ is injective.

**Problem fun.2.** Show that if $f: A \rightarrow B$ and $g: B \rightarrow C$ are both surjective, then $g \circ f: A \rightarrow C$ is surjective.

**Problem fun.3.** Suppose $f: A \rightarrow B$ and $g: B \rightarrow C$. Show that the graph of $g \circ f$ is $R_f \mid R_g$. 

![Figure 1: The composition $g \circ f$ of two functions $f$ and $g$.](image)