

story.1 Russell's Paradox (again)

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In ??, we worked with a naïve set theory. But according to a *very* naïve conception, sets are just the extensions of predicates. This naïve thought would mandate the following principle:

Naïve Comprehension. $\{x : \varphi(x)\}$ exists for any formula φ .

Tempting as this principle is, it is provably inconsistent. We saw this in ??, but the result is so important, and so straightforward, that it's worth repeating. Verbatim.

Theorem story.1 (Russell's Paradox). *There is no set $R = \{x : x \notin x\}$*

Proof. If $R = \{x : x \notin x\}$ exists, then $R \in R$ iff $R \notin R$, which is a contradiction. \square

Russell discovered this result in June 1901. (He did not, though, put the paradox in quite the form we just presented it, since he was considering Frege's set theory, as outlined in *Grundgesetze*. We will return to this in ??.) Russell wrote to Frege on June 16, 1902, explaining the inconsistency in Frege's system. For the correspondence, and a bit of background, see [Heijenoort \(1967, pp. 124–8\)](#).

It is worth emphasising that this two-line proof is a result of *pure logic*. Granted, we implicitly used a (non-logical?) axiom, Extensionality, in our notation $\{x : x \notin x\}$; for $\{x : \varphi(x)\}$ is to be *the unique* (by Extensionality) set of the φ s, if one exists. But we can avoid even the hint of Extensionality, just by stating the result as follows: *there is no set whose members are exactly the non-self-membered sets*. And this has nothing much to do with sets. As Russell himself observed, exactly similar reasoning will lead you to conclude: *no man shaves exactly the men who do not shave themselves*. Or: *no pug sniffs exactly the pugs which don't sniff themselves*. And so on. Schematically, the shape of the result is just:

$$\neg \exists x \forall z (Rzx \leftrightarrow \neg Rzz).$$

And that's just a theorem (scheme) of first-order logic. Consequently, we can't avoid Russell's Paradox just by tinkering with our set theory; it arises before we even *get* to set theory. If we're going to use (classical) first-order logic, we simply have to *accept* that there is no set $R = \{x : x \notin x\}$.

The upshot is this. If you want to accept Naïve Comprehension whilst *avoiding* inconsistency, you cannot just tinker with the *set theory*. Instead, you would have to overhaul your *logic*.

Of course, set theories with non-classical logics have been presented. But they are—to say the least—non-standard. The standard approach to Russell's Paradox is to treat it as a straightforward non-existence proof, and then to try to learn how to live with it. That is the approach we will follow.

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Bibliography

Heijenoort, Jean van. 1967. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*. Cambridge, MA: Harvard University Press.