

story.1 Appendix: Frege’s Basic Law V

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sec

In ??, we explained that Russell’s formulated his paradox as a problem for the system Frege outlined in his *Grundgesetze*. Frege’s system did not include a direct formulation of Naïve Comprehension. So, in this appendix, we will very briefly explain what Frege’s system *did* include, and how it relates to Naïve Comprehension and how it relates to Russell’s Paradox.

Frege’s system is *second-order*, and was designed to formulate the notion of an *extension of a concept*.¹ Using notation inspired by Frege, we will write $\epsilon x F(x)$ for *the extension of the concept F*. This is a device which takes a *predicate*, “*F*”, and turns it into a (first-order) *term*, “ $\epsilon x F(x)$ ”. Using this device, Frege offered the following *definition* of membership:

$$a \in b =_{\text{df}} \exists G(b = \epsilon x G(x) \wedge Ga)$$

roughly: $a \in b$ iff *a* falls under a concept whose extension is *b*. (Note that the quantifier “ $\exists G$ ” is second-order.) Frege also maintained the following principle, known as *Basic Law V*:

$$\epsilon x F(x) = \epsilon x G(x) \leftrightarrow \forall x(Fx \leftrightarrow Gx)$$

roughly: concepts have identical extensions iff they are coextensive. (Again, both “*F*” and “*G*” are in predicate position.) Now a simple principle connects membership with property-satisfaction:

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lem:Fregeextensions

Lemma story.1 (in *Grundgesetze*). $\forall F \forall a(a \in \epsilon x F(x) \leftrightarrow Fa)$

Proof. Fix *F* and *a*. Now $a \in \epsilon x F(x)$ iff $\exists G(\epsilon x F(x) = \epsilon x G(x) \wedge Ga)$ (by the definition of membership) iff $\exists G(\forall x(Fx \leftrightarrow Gx) \wedge Ga)$ (by Basic Law V) iff Fa (by elementary second-order logic). \square

And this yields Naïve Comprehension almost immediately:

Lemma story.2 (in *Grundgesetze*). $\forall F \exists s \forall a(a \in s \leftrightarrow Fa)$

Proof. Fix *F*; now **Lemma story.1** yields $\forall a(a \in \epsilon x F(x) \leftrightarrow Fa)$; so $\exists s \forall a(a \in s \leftrightarrow Fa)$ by existential generalisation. The result follows since *F* was arbitrary. \square

Russell’s Paradox follows by taking *F* as given by $\forall x(Fx \leftrightarrow x \notin x)$.

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Bibliography

Heck, Richard Kimberly. 2012. *Reading Frege’s Grundgesetze*. Oxford: Oxford University Press.

¹Strictly speaking, Frege attempts to formalize a more general notion: the “value-range” of a function. Extensions of concepts are a special case of the more general notion. See Heck (2012, pp. 8–9) for the details.