spine.1 Basic Properties of Stages

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To bring out the foundational importance of the definition of the V_αs, we will start with a few results about them.

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Lemma spine.1. *For each ordinal α:*

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1. *Each V_α is a transitive set.*

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2. *Each V_α is a* sublative *set,*[1] *i.e., $\forall A(\exists B(A \subseteq B \in V_\alpha) \rightarrow A \in V_\alpha)$.*

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3. *If $\gamma \in \alpha$, then $V_\gamma \in V_\alpha$ (and hence also $V_\gamma \subseteq V_\alpha$ by (1))*

Proof. We prove this by a (simultaneous) transfinite induction. For induction, suppose that (1)–(3) holds for each ordinal $\beta < \alpha$.

The case of $\alpha = \emptyset$ is trivial.

Suppose $\alpha = \beta^+$. To show (3), if $\gamma \in \alpha$ then $V_\gamma \subseteq V_\beta$ by hypothesis, so $V_\gamma \in \wp(V_\beta) = V_\alpha$. To show (2), suppose $A \subseteq B \in V_\alpha$ i.e., $A \subseteq B \subseteq V_\beta$; then $A \subseteq V_\beta$ so $A \in V_\alpha$. To show (1), note that if $x \in A \in V_\alpha$ we have $A \subseteq V_\beta$, so $x \in V_\beta$, so $x \subseteq V_\beta$ as V_β is transitive by hypothesis, and so $x \in V_\alpha$.

Suppose α is a limit ordinal. To show (3), if $\gamma \in \alpha$ then $\gamma \in \gamma^+ \in \alpha$, so that $V_\gamma \in V_{\gamma^+}$ by assumption, hence $V_\gamma \in \bigcup_{\beta \in \alpha} V_\beta = V_\alpha$. To show (1) and (2), just observe that a union of transitive (respectively, sublative) sets is transitive (respectively, sublative). □

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Lemma spine.2. *For each ordinal α, $V_\alpha \notin V_\alpha$.*

Proof. By transfinite induction. Evidently $V_\emptyset \notin V_\emptyset$.

If $V_{\alpha^+} \in V_{\alpha^+} = \wp(V_\alpha)$, then $V_{\alpha^+} \subseteq V_\alpha$; and since $V_\alpha \in V_{\alpha^+}$ by Lemma spine.1, we have $V_\alpha \in V_\alpha$. Conversely: if $V_\alpha \notin V_\alpha$ then $V_{\alpha^+} \notin V_{\alpha^+}$

If α is a limit and $V_\alpha \in V_\alpha = \bigcup_{\beta \in \alpha} V_\beta$, then $V_\alpha \in V_\beta$ for some $\beta \in \alpha$; but then also $V_\beta \in V_\alpha$ so that $V_\beta \in V_\beta$ by Lemma spine.1 (twice). Conversely, if $V_\beta \notin V_\beta$ for all $\beta \in \alpha$, then $V_\alpha \notin V_\alpha$. □

Corollary spine.3. *For any ordinals α, β: $\alpha \in \beta$ iff $V_\alpha \in V_\beta$*

Proof. Lemma spine.1 gives one direction. Conversely, suppose $V_\alpha \in V_\beta$. Then $\alpha \neq \beta$ by Lemma spine.2; and $\beta \notin \alpha$, for otherwise we would have $V_\beta \in V_\alpha$ and hence $V_\beta \in V_\beta$ by Lemma spine.1 (twice), contradicting Lemma spine.2. So $\alpha \in \beta$ by Trichotomy. □

All of this allows us to think of each V_α as the αth stage of the hierarchy. Here is why.

Certainly our V_αs can be thought of as being formed in an *iterative* process, for our use of ordinals tracks the notion of iteration. Moreover, if one stage is formed before the other, i.e., $V_\beta \in V_\alpha$, i.e., $\beta \in \alpha$, then our process of formation is *cumulative*, since $V_\beta \subseteq V_\alpha$. Finally, we are indeed forming *all*

[1] There's no standard terminology for "sublativity". But this seems good.

possible collections of sets that were available at any earlier stage, since any successor stage V_{α^+} is the power-set of its predecessor V_α.

In short: with $\mathbf{ZF}^-$, we are *almost* done, in articulating our vision of the cumulative-iterative hierarchy of sets. (Though, of course, we still need to justify Replacement.)

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Bibliography