## spine.1 Basic Properties of Stages

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To bring out the foundational importance of the definition of the $V_\alpha$s, we will present a few basic results about them. We start with a definition:[1]

**Definition spine.1.** The set $A$ is *potent* iff $\forall x((\exists y \in A)x \subseteq y \rightarrow x \in A)$.

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**Lemma spine.2.** *For each ordinal $\alpha$:*

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1. *Each $V_\alpha$ is transitive.*

*sth:spine:Valphabasic: Valphapotent*

2. *Each $V_\alpha$ is potent.*

*sth:spine:Valphabasic: Valphacum*

3. *If $\gamma \in \alpha$, then $V_\gamma \in V_\alpha$ (and hence also $V_\gamma \subseteq V_\alpha$ by (1))*

*Proof.* We prove this by a (simultaneous) transfinite induction. For induction, suppose that (1)–(3) holds for each ordinal $\beta < \alpha$.

The case of $\alpha = \emptyset$ is trivial.

Suppose $\alpha = \beta^+$. To show (3), if $\gamma \in \alpha$ then $V_\gamma \subseteq V_\beta$ by hypothesis, so $V_\gamma \in \wp(V_\beta) = V_\alpha$. To show (2), suppose $A \subseteq B \in V_\alpha$ i.e., $A \subseteq B \subseteq V_\beta$; then $A \subseteq V_\beta$ so $A \in V_\alpha$. To show (1), note that if $x \in A \in V_\alpha$ we have $A \subseteq V_\beta$, so $x \in V_\beta$, so $x \subseteq V_\beta$ as $V_\beta$ is transitive by hypothesis, and so $x \in V_\alpha$.

Suppose $\alpha$ is a limit ordinal. To show (3), if $\gamma \in \alpha$ then $\gamma \in \gamma^+ \in \alpha$, so that $V_\gamma \in V_{\gamma^+}$ by assumption, hence $V_\gamma \in \bigcup_{\beta\in\alpha} V_\beta = V_\alpha$. To show (1) and (2), just observe that a union of transitive (respectively, potent) sets is transitive (respectively, potent). □

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**Lemma spine.3.** *For each ordinal $\alpha$, $V_\alpha \notin V_\alpha$.*

*Proof.* By transfinite induction. Evidently $V_\emptyset \notin V_\emptyset$.

If $V_{\alpha^+} \in V_{\alpha^+} = \wp(V_\alpha)$, then $V_{\alpha^+} \subseteq V_\alpha$; and since $V_\alpha \in V_{\alpha^+}$ by Lemma spine.2, we have $V_\alpha \in V_\alpha$. Conversely: if $V_\alpha \notin V_\alpha$ then $V_{\alpha^+} \notin V_{\alpha^+}$

If $\alpha$ is a limit and $V_\alpha \in V_\alpha = \bigcup_{\beta\in\alpha} V_\beta$, then $V_\alpha \in V_\beta$ for some $\beta \in \alpha$; but then also $V_\beta \in V_\alpha$ so that $V_\beta \in V_\beta$ by Lemma spine.2 (twice). Conversely, if $V_\beta \notin V_\beta$ for all $\beta \in \alpha$, then $V_\alpha \notin V_\alpha$. □

**Corollary spine.4.** *For any ordinals $\alpha, \beta$: $\alpha \in \beta$ iff $V_\alpha \in V_\beta$*

*Proof.* Lemma spine.2 gives one direction. Conversely, suppose $V_\alpha \in V_\beta$. Then $\alpha \neq \beta$ by Lemma spine.3; and $\beta \notin \alpha$, for otherwise we would have $V_\beta \in V_\alpha$ and hence $V_\beta \in V_\beta$ by Lemma spine.2 (twice), contradicting Lemma spine.3. So $\alpha \in \beta$ by Trichotomy. □

All of this allows us to think of each $V_\alpha$ as the $\alpha$th stage of the hierarchy. Here is why.

Certainly our $V_\alpha$s can be thought of as being formed in an *iterative* process, for our use of ordinals tracks the notion of iteration. Moreover, if one stage

[1]There's no standard terminology for "potent"; this is the name used by Button (2021).

is formed before the other, i.e., $V_\beta \in V_\alpha$, i.e., $\beta \in \alpha$, then our process of formation is *cumulative*, since $V_\beta \subseteq V_\alpha$. Finally, we are indeed forming *all* possible collections of sets that were available at any earlier stage, since any successor stage $V_{\alpha^+}$ is the power-set of its predecessor $V_\alpha$.

In short: with $\mathbf{ZF}^-$, we are *almost* done, in articulating our vision of the cumulative-iterative hierarchy of sets. (Though, of course, we still need to justify Replacement.)

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## Bibliography

Button, Tim. 2021. Level theory, part 1: Axiomatizing the bare idea of a cumulative hierarchy of sets. *The Bulletin of Symbolic Logic* 27(4): 436–460.