Chapter udf

Stages and Ranks

spine.1 Defining the Stages as the V_αs

sth:spine:valpha: sec

In **??**, we defined well-orderings and the (von Neumann) ordinals. In this chapter, we will use these to characterise the hierarchy of sets *itself*. To do this, recall that in **??**, we defined the idea of successor and limit ordinals. We use these ideas in following definition:

sth:spine:valpha: defValphas

Definition spine.1.

$$\begin{aligned}
V_\emptyset &= \emptyset \\
V_{\alpha^+} &= \wp(V_\alpha) && \text{for any ordinal } \alpha \\
V_\alpha &= \bigcup_{\gamma<\alpha} V_\gamma && \text{when } \alpha \text{ is a limit ordinal}
\end{aligned}$$

This will be a definition by *transfinite recursion* on the ordinals. In this regard, we should compare this with recursive definitions of functions on the natural numbers.[1] As when dealing with natural numbers, one defines a base case and successor cases; but when dealing with ordinals, we also need to describe the behaviour of *limit* cases.

This definition of the V_αs will be an important milestone. We have informally motivated our hierarchy of sets as forming sets by *stages*. The V_αs are, in effect, just those stages. Importantly, though, this is an *internal* characterisation of the stages. Rather than suggesting a possible *model* of the theory, we will have defined the stages *within* our set theory.

spine.2 The Transfinite Recursion Theorem(s)

sth:spine:recursion: sec

The first thing we must do, though, is confirm that Definition spine.1 is a successful definition. More generally, we need to prove that any attempt to

[1] Cf. the definitions of addition, multiplication, and exponentiation in **??**.

offer a transfinite by (transfinite) recursion will succeed. That is the aim of this section.

Warning: this is tricky material. The overarching moral, though, is quite simple: Transfinite Induction plus Replacement guarantee the legitimacy of (several versions of) transfinite recursion.[2]

Definition spine.2. Let $\tau(x)$ be a term; let f be a function; let α be an ordinal. We say that f is an *α-approximation* for τ iff both $\mathrm{dom}(f) = \alpha$ and $(\forall \beta \in \alpha) f(\beta) = \tau(f{\restriction}_{\beta})$.

Lemma spine.3 (Bounded Recursion). *For any term $\tau(x)$ and any ordinal α, there is a unique α-approximation for τ.* *sth:spine:recursion:transrecursionfun*

Proof. We will show that, for any $\gamma \leq \alpha$, there is a unique γ-approximation.

We first establish uniqueness. Let g and h (respectively) be γ- and δ-approximations. A transfinite induction on their arguments shows that $g(\beta) = h(\beta)$ for any $\beta \in \mathrm{dom}(g) \cap \mathrm{dom}(h) = \gamma \cap \delta = \min(\gamma, \delta)$. So our approximations are unique (if they exist), and agree on all values.

To establish existence, we now use a simple transfinite induction (**??**) on ordinals $\delta \leq \alpha$.

The empty function is trivially an $\emptyset$-approximation.

If g is a γ-approximation, then $g \cup \{\langle \gamma^+, \tau(g) \rangle\}$ is a γ^+-approximation.

If γ is a limit ordinal and g_δ is a δ-approximation for all $\delta < \gamma$, let $g = \bigcup_{\delta \in \gamma} g_\delta$. This is a function, since our various g_δs agree on all values. And if $\delta \in \gamma$ then $g(\delta) = g_{\delta^+}(\delta) = \tau(g_{\delta^+}{\restriction}_{\delta}) = \tau(g{\restriction}_{\delta})$.

This completes the proof by transfinite induction. □

If we allow ourselves to define a *term* rather than a function, then we can remove the bound α from the previous result. In the statement and proof of the following result, when σ is a term, we let $\sigma{\restriction}_{\alpha} = \{\langle \beta, \sigma(\beta) \rangle : \beta \in \alpha\}$.

Theorem spine.4 (General Recursion). *For any term $\tau(x)$, we can explicitly define a term $\sigma(x)$, such that $\sigma(\alpha) = \tau(\sigma{\restriction}_{\alpha})$ for any ordinal α.* *sth:spine:recursion:transrecursionschema*

Proof. For each α, by Lemma spine.3 there is a unique α-approximation, f_α, for τ. Define $\sigma(\alpha)$ as $f_{\alpha^+}(\alpha)$. Now:

$$\begin{aligned}
\sigma(\alpha) &= f_{\alpha^+}(\alpha) \\
&= \tau(f_{\alpha^+}{\restriction}_{\alpha}) \\
&= \tau(\{\langle \beta, f_{\alpha^+}(\beta) \rangle : \beta \in \alpha\}) \\
&= \tau(\{\langle \beta, f_{\beta^+}(\beta) \rangle : \beta \in \alpha\}) \\
&= \tau(\sigma{\restriction}_{\alpha})
\end{aligned}$$

noting that $f_{\beta^+}(\beta) = f_{\alpha^+}(\beta)$ for all $\beta < \alpha$, as in Lemma spine.3. □

[2] A reminder: all formulas and terms can have parameters (unless explicitly stated otherwise).

Note that Theorem spine.4 is a *schema*. Crucially, we cannot expect σ to define a function, i.e., a certain kind of *set*, since then $\mathrm{dom}(\sigma)$ would be the set of all ordinals, contradicting the Burali-Forti Paradox (**??**).

It still remains to show, though, that Theorem spine.4 vindicates our definition of the V_αs. This may not be immediately obvious; but it will become apparent with a last, simple, version of transfinite recursion.

sth:spine:recursion: simplerecursionschema

Theorem spine.5 (Simple Recursion). *For any terms $\tau(x)$ and $\theta(x)$ and any set A, we can explicitly define a term $\sigma(x)$ such that:*

$$\begin{aligned} \sigma(\emptyset) &= A \\ \sigma(\alpha^+) &= \tau(\sigma(\alpha)) && \textit{for any ordinal } \alpha \\ \sigma(\alpha) &= \theta(\mathrm{ran}(\sigma{\restriction}_\alpha)) && \textit{when } \alpha \textit{ is a limit ordinal} \end{aligned}$$

Proof. We start by defining a term, $\xi(x)$, as follows:

$$\xi(x) = \begin{cases} A & \text{if } x \text{ is not a function whose} \\ & \quad \text{domain is an ordinal; otherwise:} \\ \tau(x(\alpha)) & \text{if } \mathrm{dom}(x) = \alpha^+ \\ \theta(\mathrm{ran}(x)) & \text{if } \mathrm{dom}(x) \text{ is a limit ordinal} \end{cases}$$

By Theorem spine.4, there is a term $\sigma(x)$ such that $\sigma(\alpha) = \xi(\sigma{\restriction}_\alpha)$ for every ordinal α; moreover, $\sigma{\restriction}_\alpha$ is a function with domain α. We show that σ has the required properties, by simple transfinite induction (**??**).

First, $\sigma(\emptyset) = \xi(\emptyset) = A$.

Next, $\sigma(\alpha^+) = \xi(\sigma{\restriction}_{\alpha^+}) = \tau(\sigma{\restriction}_{\alpha^+}(\alpha)) = \tau(\sigma(\alpha))$.

Last, $\sigma(\alpha) = \xi(\sigma{\restriction}_\alpha) = \theta(\mathrm{ran}(\sigma{\restriction}_\alpha))$, when α is a limit. □

Now, to vindicate Definition spine.1, just take $A = \emptyset$ and $\tau(x) = \wp(x)$ and $\theta(x) = \bigcup x$. At long last, this vindicates the definition of the V_αs!

spine.3 Basic Properties of Stages

sth:spine:Valphabasic: sec

To bring out the foundational importance of the definition of the V_αs, we will present a few basic results about them. We start with a definition:[3]

Definition spine.6. The set A is *potent* iff $\forall x((\exists y \in A)x \subseteq y \rightarrow x \in A)$.

sth:spine:Valphabasic: Valphabasicprops

Lemma spine.7. *For each ordinal α:*

sth:spine:Valphabasic: Valphatrans

1. *Each V_α is transitive.*

sth:spine:Valphabasic: Valphapotent

2. *Each V_α is potent.*

sth:spine:Valphabasic: Valphacum

3. *If $\gamma \in \alpha$, then $V_\gamma \in V_\alpha$ (and hence also $V_\gamma \subseteq V_\alpha$ by (1))*

[3]There's no standard terminology for "potent"; this is the name used by Button (2021).

Proof. We prove this by a (simultaneous) transfinite induction. For induction, suppose that (1)–(3) holds for each ordinal $\beta < \alpha$.

The case of $\alpha = \emptyset$ is trivial.

Suppose $\alpha = \beta^+$. To show (3), if $\gamma \in \alpha$ then $V_\gamma \subseteq V_\beta$ by hypothesis, so $V_\gamma \in \wp(V_\beta) = V_\alpha$. To show (2), suppose $A \subseteq B \in V_\alpha$ i.e., $A \subseteq B \subseteq V_\beta$; then $A \subseteq V_\beta$ so $A \in V_\alpha$. To show (1), note that if $x \in A \in V_\alpha$ we have $A \subseteq V_\beta$, so $x \in V_\beta$, so $x \subseteq V_\beta$ as V_β is transitive by hypothesis, and so $x \in V_\alpha$.

Suppose α is a limit ordinal. To show (3), if $\gamma \in \alpha$ then $\gamma \in \gamma^+ \in \alpha$, so that $V_\gamma \in V_{\gamma^+}$ by assumption, hence $V_\gamma \in \bigcup_{\beta \in \alpha} V_\beta = V_\alpha$. To show (1) and (2), just observe that a union of transitive (respectively, potent) sets is transitive (respectively, potent). □

Lemma spine.8. *For each ordinal α, $V_\alpha \notin V_\alpha$.* sth:spine:Valphabasic:Valphanotref

Proof. By transfinite induction. Evidently $V_\emptyset \notin V_\emptyset$.

If $V_{\alpha^+} \in V_{\alpha^+} = \wp(V_\alpha)$, then $V_{\alpha^+} \subseteq V_\alpha$; and since $V_\alpha \in V_{\alpha^+}$ by Lemma spine.7, we have $V_\alpha \in V_\alpha$. Conversely: if $V_\alpha \notin V_\alpha$ then $V_{\alpha^+} \notin V_{\alpha^+}$

If α is a limit and $V_\alpha \in V_\alpha = \bigcup_{\beta \in \alpha} V_\beta$, then $V_\alpha \in V_\beta$ for some $\beta \in \alpha$; but then also $V_\beta \in V_\alpha$ so that $V_\beta \in V_\beta$ by Lemma spine.7 (twice). Conversely, if $V_\beta \notin V_\beta$ for all $\beta \in \alpha$, then $V_\alpha \notin V_\alpha$. □

Corollary spine.9. *For any ordinals α, β: $\alpha \in \beta$ iff $V_\alpha \in V_\beta$*

Proof. Lemma spine.7 gives one direction. Conversely, suppose $V_\alpha \in V_\beta$. Then $\alpha \neq \beta$ by Lemma spine.8; and $\beta \notin \alpha$, for otherwise we would have $V_\beta \in V_\alpha$ and hence $V_\beta \in V_\beta$ by Lemma spine.7 (twice), contradicting Lemma spine.8. So $\alpha \in \beta$ by Trichotomy. □

All of this allows us to think of each V_α as the αth stage of the hierarchy. Here is why.

Certainly our V_αs can be thought of as being formed in an *iterative* process, for our use of ordinals tracks the notion of iteration. Moreover, if one stage is formed before the other, i.e., $V_\beta \in V_\alpha$, i.e., $\beta \in \alpha$, then our process of formation is *cumulative*, since $V_\beta \subseteq V_\alpha$. Finally, we are indeed forming *all* possible collections of sets that were available at any earlier stage, since any successor stage V_{α^+} is the power-set of its predecessor V_α.

In short: with $\mathbf{ZF}^-$, we are *almost* done, in articulating our vision of the cumulative-iterative hierarchy of sets. (Though, of course, we still need to justify Replacement.)

spine.4 Foundation

We are only *almost* done—and not *quite* finished—because nothing in $\mathbf{ZF}^-$ guarantees that *every* set is in some V_α, i.e., that every set is formed at some stage. sth:spine:foundation:sec

Now, there is a fairly straightforward (mathematical) sense in which we don't *care* whether there are sets outside the hierarchy. (If there are any there, we can simply ignore them.) But we have motivated our *concept* of set with the thought that every set is formed at some stage (see *Stages-are-key* in **??**). So we will want to preclude the possibility of sets which fall outside of the hierarchy. Accordingly, we must add a new axiom, which ensures that every set occurs somewhere in the hierarchy.

Since the V_αs are our stages, we might simply consider adding the following as an axiom:

Regularity. $\forall A \exists \alpha\, A \subseteq V_\alpha$

This would be a perfectly reasonable approach. However, for reasons that will be explained in the next section, we will instead adopt an alternative axiom:

Axiom (Foundation). $(\forall A \neq \emptyset)(\exists B \in A) A \cap B = \emptyset$.

With some effort, we can show (in $\mathbf{ZF}^-$) that Foundation entails Regularity:

Definition spine.10. For each set A, let:

$$\begin{aligned} \mathrm{cl}_0(A) &= A, \\ \mathrm{cl}_{n+1}(A) &= \bigcup \mathrm{cl}_n(A), \\ \mathrm{trcl}(A) &= \bigcup_{n<\omega} \mathrm{cl}_n(A). \end{aligned}$$

We call $\mathrm{trcl}(A)$ the *transitive closure* of A.

The name "transitive closure" is apt:

sth:spine:foundation: subsetoftrcl

Proposition spine.11. *$A \subseteq \mathrm{trcl}(A)$ and $\mathrm{trcl}(A)$ is a transitive set.*

Proof. Evidently $A = \mathrm{cl}_0(A) \subseteq \mathrm{trcl}(A)$. And if $x \in b \in \mathrm{trcl}(A)$, then $b \in \mathrm{cl}_n(A)$ for some n, so $x \in \mathrm{cl}_{n+1}(A) \subseteq \mathrm{trcl}(A)$. □

sth:spine:foundation: lem:TransitiveWellFounded

Lemma spine.12. *If A is a transitive set, then there is some α such that $A \subseteq V_\alpha$.*

Proof. Recalling the definition of "$\mathrm{lsub}(X)$" from **??**, define two sets:

$$\begin{aligned} D &= \{x \in A : \forall \delta\ x \nsubseteq V_\delta\} \\ \alpha &= \mathrm{lsub}\{\delta : (\exists x \in A)(x \subseteq V_\delta \wedge (\forall \gamma \in \delta) x \nsubseteq V_\gamma)\} \end{aligned}$$

Suppose $D = \emptyset$. So if $x \in A$, then there is some δ such that $x \subseteq V_\delta$ and, by the well-ordering of the ordinals, $(\forall \gamma \in \delta) x \nsubseteq V_\gamma$; hence $\delta \in \alpha$ and so $x \in V_\alpha$ by Lemma spine.7. Hence $A \subseteq V_\alpha$, as required.

So it suffices to show that $D = \emptyset$. For reductio, suppose otherwise. By Foundation, there is some $B \in D \subseteq A$ such that $D \cap B = \emptyset$. If $x \in B$ then $x \in A$, since A is transitive, and since $x \notin D$, it follows that $\exists \delta\ x \subseteq V_\delta$. So now let

$$\beta = \text{lsub}\{\delta : (\exists x \in b)(x \subseteq V_\delta \wedge (\forall \gamma < \delta) x \nsubseteq V_\gamma)\}.$$

As before, $B \subseteq V_\beta$, contradicting the claim that $B \in D$. □

Theorem spine.13. *Regularity holds.* sth:spine:foundation: zfentailsregularity

Proof. Fix A; now $A \subseteq \text{trcl}(A)$ by Proposition spine.11, which is transitive. So there is some α such that $A \subseteq \text{trcl}(A) \subseteq V_\alpha$ by Lemma spine.12 □

These results show that $\mathbf{ZF}^-$ proves the conditional *Foundation* $\Rightarrow$ *Regularity*. In Proposition spine.22, we will show that $\mathbf{ZF}^-$ proves *Regularity* $\Rightarrow$ *Foundation*. As such, Foundation and Regularity are *equivalent* (modulo $\mathbf{ZF}^-$). But this means that, given $\mathbf{ZF}^-$, we can justify Foundation by noting that it is equivalent to Regularity. And we can justify Regularity immediately on the basis of *Stages-are-key*.

spine.5 Z and ZF: A Milestone

With Foundation, we reach another important milestone. We have considered theories $\mathbf{Z}^-$ and $\mathbf{ZF}^-$, which we said were certain theories "minus" a certain something. That certain something is Foundation. So: sth:spine:zf: sec

Definition spine.14. The theory $\mathbf{Z}$ adds Foundation to $\mathbf{Z}^-$. So its axioms are Extensionality, Union, Pairs, Powersets, Infinity, Foundation, and all instances of the Separation scheme.

The theory $\mathbf{ZF}$ adds Foundation to $\mathbf{ZF}^-$. Otherwise put, $\mathbf{ZF}$ adds all instances of Replacement to $\mathbf{Z}$.

Still, one question might have occurred to you. If Regularity is equivalent over $\mathbf{ZF}^-$ to Foundation, and Regularity's justification is clear, why bother to go around the houses, and take Foundation as our basic axiom, rather than Regularity?

Setting aside historical reasons (to do with who formulated what and when), the basic reason is that Foundation can be presented without employing the definition of the V_αs. That definition relied upon all of the work of section spine.2: we needed to prove Transfinite Recursion, to show that it was justified. But our proof of Transfinite Recursion employed *Replacement*. So, whilst Foundation and Regularity are equivalent modulo $\mathbf{ZF}^-$, they are not equivalent modulo $\mathbf{Z}^-$.

Indeed, the matter is more drastic than this simple remark suggests. Though it goes well beyond this book's remit, it turns out that both $\mathbf{Z}^-$ and $\mathbf{Z}$ are too weak to define the V_αs. So, if you are working only in $\mathbf{Z}$, then Regularity (as we have formulated it) does not even make *sense*. This is why our official axiom is Foundation, rather than Regularity.

From now on, we will work in **ZF** (unless otherwise stated), without any further comment.

spine.6 Rank

sth:spine:rank: sec

Now that we have defined the stages as the V_α's, and we know that every set is a subset of some stage, we can define the *rank* of a set. Intuitively, the rank of A is the first moment at which A is formed. More precisely:

sth:spine:rank: defnsetrank

Definition spine.15. For each set A, $\text{rank}(A)$ is the least ordinal α such that $A \subseteq V_\alpha$.

sth:spine:rank: ranksexist

Proposition spine.16. $\text{rank}(A)$ *exists, for any* A.

Proof. Left as an exercise. □

Problem spine.1. Prove Proposition spine.16.

The well-ordering of ranks allows us to prove some important results:

sth:spine:rank: valphalowerrank

Proposition spine.17. *For any ordinal* α, $V_\alpha = \{x : \text{rank}(x) \in \alpha\}$.

Proof. If $\text{rank}(x) \in \alpha$ then $x \subseteq V_{\text{rank}(x)} \in V_\alpha$, so $x \in V_\alpha$ as V_α is potent (invoking Lemma spine.7 multiple times). Conversely, if $x \in V_\alpha$ then $x \subseteq V_\alpha$, so $\text{rank}(x) \leq \alpha$; now a simple transfinite induction shows that $x \notin V_\alpha$. □

Problem spine.2. Complete the simple transfinite induction mentioned in Proposition spine.17.

sth:spine:rank: rankmemberslower

Proposition spine.18. *If* $B \in A$*, then* $\text{rank}(B) \in \text{rank}(A)$.

Proof. $A \subseteq V_{\text{rank}(A)} = \{x : \text{rank}(x) \in \text{rank}(A)\}$ by Proposition spine.17. □

Using this fact, we can establish a result which allows us to prove things about *all sets* by a form of induction:

Theorem spine.19 ($\in$-Induction Scheme). *For any formula* φ*:*

$$\forall A((\forall x \in A)\varphi(x) \rightarrow \varphi(A)) \rightarrow \forall A\varphi(A).$$

Proof. We will prove the contrapositive. So, suppose $\neg\forall A\varphi(A)$. By Transfinite Induction (**??**), there is some non-φ of least possible rank; i.e. some A such that $\neg\varphi(A)$ and $\forall x(\text{rank}(x) \in \text{rank}(A) \rightarrow \varphi(x))$. Now if $x \in A$ then $\text{rank}(x) \in \text{rank}(A)$, by Proposition spine.18, so that $\varphi(x)$; i.e. $(\forall x \in A)\varphi(x) \wedge \neg\varphi(A)$. □

Here is an informal way to gloss this powerful result. Say that φ is *hereditary* iff whenever every element of a set is φ, the set itself is φ. Then $\in$-Induction tells you the following: if φ is hereditary, every set is φ.

To wrap up the discussion of ranks (for now), we'll prove a few claims which we have foreshadowed a few times.

Proposition spine.20. $\text{rank}(A) = \text{lsub}_{x\in A}\,\text{rank}(x)$. *sth:spine:rank:ranksupstrict*

Proof. Let $\alpha = \text{lsub}_{x\in A}\,\text{rank}(x)$. By Proposition spine.18, $\alpha \leq \text{rank}(A)$. But if $x \in A$ then $\text{rank}(x) \in \alpha$, so that $x \in V_\alpha$ by Proposition spine.17, and hence $A \subseteq V_\alpha$, i.e., $\text{rank}(A) \leq \alpha$. Hence $\text{rank}(A) = \alpha$. □

Corollary spine.21. *For any ordinal α, $\text{rank}(\alpha) = \alpha$.* *sth:spine:rank:ordsetrankalpha*

Proof. Suppose for transfinite induction that $\text{rank}(\beta) = \beta$ for all $\beta \in \alpha$. Now $\text{rank}(\alpha) = \text{lsub}_{\beta\in\alpha}\,\text{rank}(\beta) = \text{lsub}_{\beta\in\alpha}\,\beta = \alpha$ by Proposition spine.20. □

Finally, here is a quick proof of the result promised at the end of section spine.4, that $\mathbf{ZF}^-$ proves the conditional *Regularity* $\Rightarrow$ *Foundation*. (Note that the notion of "rank" and Proposition spine.18 are available for use in this proof since—as mentioned at the start of this section—they can be presented using $\mathbf{ZF}^- +$ Regularity.)

Proposition spine.22 (working in $\mathbf{ZF}^- +$ Regularity). *Foundation holds.* *sth:spine:rank:zfminusregularityfoundation*

Proof. Fix $A \neq \emptyset$, and some $B \in A$ of least possible rank. If $c \in B$ then $\text{rank}(c) \in \text{rank}(B)$ by Proposition spine.18, so that $c \notin A$ by choice of B. □

Photo Credits

Bibliography

Button, Tim. 2021. Level theory, part 1: Axiomatizing the bare idea of a cumulative hierarchy of sets. *The Bulletin of Symbolic Logic* 27(4): 436–460.