spine.1 The Transfinite Recursion Theorem(s)

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The first thing we must do, though, is confirm that **??** is a successful definition. More generally, we need to prove that any attempt to offer a transfinite by (transfinite) recursion will succeed. That is the aim of this section.

Warning: this is tricky material. The overarching moral, though, is quite simple: Transfinite Induction plus Replacement guarantee the legitimacy of (several versions of) transfinite recursion.[1]

Definition spine.1. Let $\tau(x)$ be a term; let f be a function; let α be an ordinal. We say that f is an *α-approximation* for τ iff both $\mathrm{dom}(f) = \alpha$ and $(\forall \beta \in \alpha) f(\beta) = \tau(f{\restriction}_{\beta})$.

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Lemma spine.2 (Bounded Recursion). *For any term $\tau(x)$ and any ordinal α, there is a unique α-approximation for τ.*

Proof. We will show that, for any $\gamma \leq \alpha$, there is a unique γ-approximation.

We first establish uniqueness. Let g and h (respectively) be γ- and δ-approximations. A transfinite induction on their arguments shows that $g(\beta) = h(\beta)$ for any $\beta \in \mathrm{dom}(g) \cap \mathrm{dom}(h) = \gamma \cap \delta = \min(\gamma, \delta)$. So our approximations are unique (if they exist), and agree on all values.

To establish existence, we now use a simple transfinite induction (**??**) on ordinals $\delta \leq \alpha$.

The empty function is trivially an $\emptyset$-approximation.

If g is a γ-approximation, then $g \cup \{\langle \gamma^+, \tau(g)\rangle\}$ is a γ^+-approximation.

If γ is a limit ordinal and g_δ is a δ-approximation for all $\delta < \gamma$, let $g = \bigcup_{\delta \in \gamma} g_\delta$. This is a function, since our various g_δs agree on all values. And if $\delta \in \gamma$ then $g(\delta) = g_{\delta^+}(\delta) = \tau(g_{\delta^+}{\restriction}_{\delta}) = \tau(g{\restriction}_{\delta})$.

This completes the proof by transfinite induction. □

If we allow ourselves to define a *term* rather than a function, then we can remove the bound α from the previous result. In the statement and proof of the following result, when σ is a term, we let $\sigma{\restriction}_{\alpha} = \{\langle \beta, \sigma(\beta)\rangle : \beta \in \alpha\}$.

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Theorem spine.3 (General Recursion). *For any term $\tau(x)$, we can explicitly define a term $\sigma(x)$, such that $\sigma(\alpha) = \tau(\sigma{\restriction}_{\alpha})$ for any ordinal α.*

Proof. For each α, by Lemma spine.2 there is a unique α-approximation, f_α, for τ. Define $\sigma(\alpha)$ as $f_{\alpha^+}(\alpha)$. Now:

$$\begin{aligned}
\sigma(\alpha) &= f_{\alpha^+}(\alpha)\\
&= \tau(f_{\alpha^+}{\restriction}_{\alpha})\\
&= \tau(\{\langle \beta, f_{\alpha^+}(\beta)\rangle : \beta \in \alpha\})\\
&= \tau(\{\langle \beta, f_{\beta^+}(\beta)\rangle : \beta \in \alpha\})\\
&= \tau(\sigma{\restriction}_{\alpha})
\end{aligned}$$

[1] A reminder: all formulas and terms can have parameters (unless explicitly stated otherwise).

noting that $f_{\beta^+}(\beta) = f_{\alpha^+}(\beta)$ for all $\beta < \alpha$, as in Lemma spine.2. $\square$

Note that Theorem spine.3 is a *schema*. Crucially, we cannot expect σ to define a function, i.e., a certain kind of *set*, since then $\mathrm{dom}(\sigma)$ would be the set of all ordinals, contradicting the Burali-Forti Paradox (**??**).

It still remains to show, though, that Theorem spine.3 vindicates our definition of the V_αs. This may not be immediately obvious; but it will become apparent with a last, simple, version of transfinite recursion.

Theorem spine.4 (Simple Recursion). *For any terms $\tau(x)$ and $\theta(x)$ and any set A, we can explicitly define a term $\sigma(x)$ such that:* sth:spine:recursion:simplerecursionschema

$$\begin{aligned} \sigma(\emptyset) &= A \\ \sigma(\alpha^+) &= \tau(\sigma(\alpha)) && \text{for any ordinal } \alpha \\ \sigma(\alpha) &= \theta(\mathrm{ran}(\sigma{\restriction}_\alpha)) && \text{when } \alpha \text{ is a limit ordinal} \end{aligned}$$

Proof. We start by defining a term, $\xi(x)$, as follows:

$$\xi(x) = \begin{cases} A & \text{if } x \text{ is not a function whose} \\ & \quad \text{domain is an ordinal; otherwise:} \\ \tau(x(\alpha)) & \text{if } \mathrm{dom}(x) = \alpha^+ \\ \theta(\mathrm{ran}(x)) & \text{if } \mathrm{dom}(x) \text{ is a limit ordinal} \end{cases}$$

By Theorem spine.3, there is a term $\sigma(x)$ such that $\sigma(\alpha) = \xi(\sigma{\restriction}_\alpha)$ for every ordinal α; moreover, $\sigma{\restriction}_\alpha$ is a function with domain α. We show that σ has the required properties, by simple transfinite induction (**??**).

First, $\sigma(\emptyset) = \xi(\emptyset) = A$.

Next, $\sigma(\alpha^+) = \xi(\sigma{\restriction}_{\alpha^+}) = \tau(\sigma{\restriction}_{\alpha^+}(\alpha)) = \tau(\sigma(\alpha))$.

Last, $\sigma(\alpha) = \xi(\sigma{\restriction}_\alpha) = \theta(\mathrm{ran}(\sigma{\restriction}_\alpha))$, when α is a limit. $\square$

Now, to vindicate **??**, just take $A = \emptyset$ and $\tau(x) = \wp(x)$ and $\theta(x) = \bigcup x$. At long last, this vindicates the definition of the V_αs!

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Bibliography