The first thing we must do, though, is confirm that \( \tau \) is a successful definition. More generally, we need to prove that any attempt to offer a transfinite by (transfinite) recursion will succeed. That is the aim of this section.

**Warning:** this is tricky material. The overarching moral, though, is quite simple: Transfinite Induction plus Replacement guarantee the legitimacy of (several versions of) transfinite recursion.¹

**Definition spine.1.** Let \( \tau(x) \) be a term; let \( f \) be a function; let \( \alpha \) be an ordinal. We say that \( f \) is an \( \alpha \)-approximation for \( \tau \) iff both \( \text{dom}(f) = \alpha \) and \((\forall \beta \in \alpha) f(\beta) = \tau(f|_\beta)\).

**Lemma spine.2 (Bounded Recursion).** For any term \( \tau(x) \) and any ordinal \( \alpha \), there is a unique \( \alpha \)-approximation for \( \tau \).

**Proof.** We will show that, for any \( \gamma \leq \alpha \), there is a unique \( \gamma \)-approximation.

We first establish uniqueness. Let \( g \) and \( h \) (respectively) be \( \gamma \)- and \( \delta \)-approximations. A transfinite induction on their arguments shows that \( g(\beta) = h(\beta) \) for any \( \beta \in \text{dom}(g) \cap \text{dom}(h) = \gamma \cap \delta = \min(\gamma, \delta) \). So our approximations are unique (if they exist), and agree on all values.

To establish existence, we now use a simple transfinite induction (??) on ordinals \( \delta \leq \alpha \).

The empty function is trivially an \( \emptyset \)-approximation.

If \( g \) is a \( \gamma \)-approximation, then \( g \cup \{ (\gamma^+, \tau(g)) \} \) is a \( \gamma^+ \)-approximation.

If \( \gamma \) is a limit ordinal and \( g_\delta \) is a \( \delta \)-approximation for all \( \delta < \gamma \), let \( g = \bigcup_{\delta \leq \gamma} g_\delta \). This is a function, since our various \( g_\delta \)s agree on all values. And if \( \delta \in \gamma \) then \( g(\delta) = g_\delta(\delta) = \tau(g_\delta|_\delta) = \tau(g|_\delta) \).

This completes the proof by transfinite induction.

If we allow ourselves to define a term rather than a function, then we can remove the bound \( \alpha \) from the previous result. In the statement and proof of the following result, when \( \sigma \) is a term, we let \( \sigma|_\alpha = \{ (\beta, \sigma(\beta)) : \beta \in \alpha \} \).

**Theorem spine.3 (General Recursion).** For any term \( \tau(x) \), we can explicitly define a term \( \sigma(x) \), such that \( \sigma(\alpha) = \tau(\sigma|_\alpha) \) for any ordinal \( \alpha \).

**Proof.** For each \( \alpha \), by Lemma spine.2 there is a unique \( \alpha \)-approximation, \( f_\alpha \), for \( \tau \). Define \( \sigma(\alpha) \) as \( f_\alpha^+(\alpha) \). Now:

\[
\sigma(\alpha) = f_\alpha^+(\alpha) = \tau(f_\alpha^+|_\alpha) = \tau(\{ (\beta, f_\alpha^+(\beta)) : \beta \in \alpha \}) = \tau(\{ (\beta, f_\beta^+(\beta)) : \beta \in \alpha \}) = \tau(\sigma|_\alpha)
\]

¹A reminder: all formulas and terms can have parameters (unless explicitly stated otherwise).
noting that $f_{\beta^+}(\beta) = f_{\alpha^+}(\beta)$ for all $\beta < \alpha$, as in Lemma spine.2.

Note that Theorem spine.3 is a schema. Crucially, we cannot expect $\sigma$ to define a function, i.e., a certain kind of set, since then dom($\sigma$) would be the set of all ordinals, contradicting the Burali-Forti Paradox (??).

It still remains to show, though, that Theorem spine.3 vindicates our definition of the $V_\alpha$s. This may not be immediately obvious; but it will become apparent with a last, simple, version of transfinite recursion.

**Theorem spine.4 (Simple Recursion).** For any terms $\tau(x)$ and $\theta(x)$ and any set $A$, we can explicitly define a term $\sigma(x)$ such that:

- $\sigma(\emptyset) = A$
- $\sigma(\alpha^+) = \tau(\sigma(\alpha))$ for any ordinal $\alpha$
- $\sigma(\alpha) = \theta(\text{ran}(\sigma|_\alpha))$ when $\alpha$ is a limit ordinal

**Proof.** We start by defining a term, $\xi(x)$, as follows:

$$
\xi(x) = \begin{cases} 
A & \text{if } x \text{ is not a function whose domain is an ordinal; otherwise:} \\
\tau(x(\alpha)) & \text{if } \text{dom}(x) = \alpha^+
\end{cases}
$$

By Theorem spine.3, there is a term $\sigma(x)$ such that $\sigma(\alpha) = \xi(\sigma|_\alpha)$ for every ordinal $\alpha$; moreover, $\sigma|_\alpha$ is a function with domain $\alpha$. We show that $\sigma$ has the required properties, by simple transfinite induction (??).

First, $\sigma(\emptyset) = \xi(\emptyset) = A$.

Next, $\sigma(\alpha^+) = \xi(\sigma|_{\alpha^+}) = \tau(\sigma|_{\alpha^+}(\alpha)) = \tau(\sigma(\alpha))$.

Last, $\sigma(\alpha) = \xi(\sigma|_\alpha) = \theta(\text{ran}(\sigma|_\alpha))$, when $\alpha$ is a limit. \square

Now, to vindicate ??, just take $A = \emptyset$ and $\tau(x) = \wp(x)$ and $\theta(x) = \bigcup x$. At long last, this vindicates the definition of the $V_\alpha$s!

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**Bibliography**