spine.1 The Transfinite Recursion Theorem(s)

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The first thing we must do, though, is confirm that **??** is a successful definition. More generally, we need to prove that any attempt to offer a transfinite by (transfinite) recursion will succeed. That is the aim of this section.

Warning: this is very tricky material. The overarching moral, though, is quite simple: Transfinite Induction plus Replacement guarantee the legitimacy of (several versions of) transfinite recursion.

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Theorem spine.1 (Bounded Recursion). *For any term $\tau(x)$ and any ordinal α,*[1] *there is a unique function f with domain α such that* $(\forall \beta \in \alpha) f(\beta) = \tau(f{\restriction}_{\beta})$

Proof. We will show that, for any $\delta \leq \alpha$, there is a unique g_δ with domain δ such that $(\forall \beta \in \delta) g(\beta) = \tau(g{\restriction}_{\beta})$.

We first establish uniqueness. Given g_{δ_1} and g_{δ_2}, a transfinite induction on their arguments shows that $g(\beta) = h(\beta)$ for any $\beta \in \mathrm{dom}(g) \cap \mathrm{dom}(h) = \delta_1 \cap \delta_2 = \min(\delta_1, \delta_2)$. So our functions are unique (if they exist), and agree on all values.

To establish existence, we now use a simple transfinite induction (**??**) on ordinals $\delta \leq \alpha$.

Let $g_\emptyset = \emptyset$; this trivially behaves correctly.

Given g_δ, let $g_{\delta^+} = g_\delta \cup \{\langle \delta, \tau(g_\delta) \rangle\}$. This behaves correctly as $g_{\delta^+}{\restriction}_{\delta} = g_\delta$.

Given g_γ for all $\gamma \leq \delta$ with δ a limit ordinal, let $g_\delta = \bigcup_{\gamma \in \delta} g_\gamma$. This is a function, since our various g_β's agree on all values. And if $\beta \in \delta$ then $g_\delta(\beta) = g_{\beta^+}(\beta) = \tau(g_{\beta^+}{\restriction}_{\beta}) = \tau(g_\delta{\restriction}_{\beta})$.

This completes the proof by transfinite induction. Now just let $f = g_\alpha$. □

If we allow ourselves to define a *term* rather than a function, then we can remove the bound α from the previous result. (In the statement and proof of this result, when σ is a term, we let $\sigma{\restriction}_{\alpha} = \{\langle \gamma, \sigma(\gamma) \rangle : \gamma \in \alpha\}$.)

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Theorem spine.2 (General Recursion). *For any term $\tau(x)$ we can explicitly define a term $\sigma(x)$,*[2] *such that $\sigma(\alpha) = \tau(\sigma{\restriction}_{\alpha})$ for any ordinal α.*

Proof. For each α, by Theorem spine.1 are unique α^+-approximations, f_{α^+}, and:

$$f_{\alpha^+}(\alpha) = \tau(f_{\alpha^+}{\restriction}_{\alpha}) = \tau(\{\langle \gamma, f_{\alpha^+}(\gamma) \rangle : \gamma \in \alpha\}).$$

So define $\sigma(\alpha)$ as $f_{\alpha^+}(\alpha)$. Repeating the induction of Theorem spine.1, but without the upper bound, this is well-defined. □

Note that these results are *schemas*. Crucially, we cannot expect σ to define a function, i.e., a certain kind of *set*, since then $\mathrm{dom}(\sigma)$ would be the set of all ordinals, contradicting the Burali-Forti Paradox (**??**).

[1] The term may have parameters.
[2] Both terms may have parameters.

It still remains to show, though, that Theorem spine.2 vindicates our definition of the V_αs. This may not be immediately obvious; but it will become apparent with a last, simple, version of transfinite recursion.

Theorem spine.3 (Simple Recursion). *For any terms $\tau_1(x)$ and $\tau_2(x)$ and any set A, we can explicitly define a term $\sigma(x)$ such that:*[3] sth:spine:recursion: simplerecursionschema

$$\begin{aligned} \sigma(\emptyset) &= A \\ \sigma(\alpha^+) &= \tau_1(\sigma(\alpha)) && \textit{for any ordinal } \alpha \\ \sigma(\alpha) &= \tau_2(\mathrm{ran}(\sigma{\restriction}_\alpha)) && \textit{when } \alpha \textit{ is a limit ordinal} \end{aligned}$$

Proof. We start by defining a term, $\xi(x)$, as follows:

$$\xi(x) = \begin{cases} A & \text{if } x \text{ is not a function whose domain is an ordinal;} \\ & \quad\text{otherwise:} \\ \tau_1(x(\alpha)) & \text{if } \mathrm{dom}(x) = \alpha^+ \\ \tau_2(\mathrm{ran}(x)) & \text{if } \mathrm{dom}(x) \text{ is a limit ordinal} \end{cases}$$

By Theorem spine.2, there is a term $\sigma(x)$ such that $\sigma(\alpha) = \xi(\sigma{\restriction}_\alpha)$ for every ordinal α; moreover, $\sigma{\restriction}_\alpha$ is a function with domain α. We show that σ has the required properties, by simple transfinite induction (**??**).

First, $\sigma(\emptyset) = \xi(\emptyset) = A$.

Next, $\sigma(\alpha^+) = \xi(\sigma{\restriction}_{\alpha^+}) = \tau_1(\sigma{\restriction}_{\alpha^+}(\alpha)) = \tau_1(\sigma(\alpha))$.

Finally, if α is a limit ordinal, $\sigma(\alpha) = \xi(\sigma{\restriction}_\alpha) = \tau_2(\mathrm{ran}(\sigma{\restriction}_\alpha))$. □

Now, to vindicate **??**, just take $A = \emptyset$ and $\tau_1(x) = \wp(x)$ and $\tau_2(x) = \bigcup x$. So we have vindicated the definition of the V_αs!

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Bibliography

[3] The terms may have parameters.