spine.1 Rank

sth:spine:rank:sec

Now that we have defined the stages as the V_α's, and we know that every set is a subset of some stage, we can define the *rank* of a set. Intuitively, the rank of A is the first moment at which A is formed. More precisely:

sth:spine:rank:defnsetrank

Definition spine.1. For each set A, $\text{rank}(A)$ is the least α such that $A \subseteq V_\alpha$.[1]

The well-ordering of ranks allows us to prove some important results:

sth:spine:rank:valphalowerrank

Proposition spine.2. *For any ordinal* α, $V_\alpha = \{x : \text{rank}(x) \in \alpha\}$.

Proof. If $\text{rank}(x) \in \alpha$ then $x \subseteq V_{\text{rank}(x)} \in V_\alpha$, so $x \in V_\alpha$ as V_α is sublative (invoking **??** multiple times). Conversely, by definition of "rank" and Trichotomy on ordinals, if $\text{rank}(x) \notin \alpha$, then $x \nsubseteq V_\beta$ for any $\beta \in \alpha$; and a simple transfinite induction on ordinals up to α shows that $x \notin V_\alpha$. □

sth:spine:rank:rankmemberslower

Proposition spine.3. *If* $B \in A$, *then* $\text{rank}(B) \in \text{rank}(A)$.

Proof. $A \subseteq V_{\text{rank}(A)} = \{x : \text{rank}(x) \in \text{rank}(A)\}$ by Proposition spine.2. □

Using this fact, we can establish a result which allows us to prove things about *all sets* by a form of induction:

Theorem spine.4 ($\in$-Induction Scheme). *For any formula* φ:[2]

$$\forall A((\forall x \in A)\varphi(x) \rightarrow \varphi(A)) \rightarrow \forall A \varphi(A).$$

Proof. We will prove the contrapositive. So, suppose $\neg\forall A\varphi(A)$. Since every set has a rank, Transfinite Induction (**??**) tells us that there is a non-φ of least possible rank. That is: there is some A such that $\neg\varphi(A)$ and $\forall x(\text{rank}(x) \in \text{rank}(A) \rightarrow \varphi(x))$. Now if $x \in A$ then $\text{rank}(x) \in \text{rank}(A)$, by Proposition spine.3. So $(\forall x \in A)\varphi(x) \wedge \neg\varphi(A)$, falsifying the antecedent. □

Here is an informal way to gloss this powerful result. Say that φ is *hereditary* iff whenever every elements of a set is φ, the set itself is φ. Then $\in$-Induction tells you the following: if φ is hereditary, every set is φ.

To wrap up the discussion of ranks (for now), we'll prove a few claims which we have foreshadowed a few times.

sth:spine:rank:ranksupstrict

Proposition spine.5. $\text{rank}(A) = \text{lsub}_{x \in A}\, \text{rank}(x)$.

Proof. Let $\alpha = \text{lsub}_{x \in A}\, \text{rank}(x)$. By Proposition spine.3, $\alpha \leq \text{rank}(A)$. But if $x \in A$ then $\text{rank}(x) \in \alpha$, so that $x \in V_\alpha$, and hence $A \subseteq V_\alpha$, i.e., $\text{rank}(A) \leq \alpha$. Hence $\text{rank}(A) = \alpha$. □

[1] Some books define $\text{rank}(A)$ as the least α such that $A \in V_\alpha$. Since $A \subseteq V_\alpha \leftrightarrow A \in V_{\alpha+1}$, this is essentially just a notational difference.

[2] Which may have parameters

Corollary spine.6. *For any ordinal* α*,* $\text{rank}(\alpha) = \alpha$*.* sth:spine:rank: ordsetrankalpha

Proof. Suppose for transfinite induction that $\text{rank}(\beta) = \beta$ for all $\beta \in \alpha$. Now $\text{rank}(\alpha) = \text{lsub}_{\beta \in \alpha} \text{rank}(\beta) = \text{lsub}_{\beta \in \alpha} \beta = \alpha$ by Proposition spine.5. □

Finally, here is a quick proof of the result promised at the end of **??**, that $\mathbf{ZF}^-$ proves the conditional *Regularity* $\Rightarrow$ *Foundation*. (Note that the notion of "rank" and Proposition spine.3 are available for use in this proof since—as mentioned at the start of this section—they can be presented using $\mathbf{ZF}^- +$ Regularity.)

Proposition spine.7 (working in $\mathbf{ZF}^- +$ Regularity). *Foundation holds.* sth:spine:rank: zfminusregularityfoundation

Proof. Fix $A \neq \emptyset$, and some $B \in A$ of least possible rank. If $c \in B$ then $\text{rank}(c) \in \text{rank}(B)$ by Proposition spine.3, so that $c \notin A$ by choice of B. □

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Bibliography