spine.1 Rank

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sec

Now that we have defined the stages as the V_α's, and we know that every set is a subset of some stage, we can define the *rank* of a set. Intuitively, the rank of A is the first moment at which A is formed. More precisely:

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defnsetrank

Definition spine.1. For each set A, $\operatorname{rank}(A)$ is the least ordinal α such that $A \subseteq V_\alpha$.

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ranksexist

Proposition spine.2. $\operatorname{rank}(A)$ *exists, for any* A.

Proof. Left as an exercise. □

Problem spine.1. Prove Proposition spine.2.

The well-ordering of ranks allows us to prove some important results:

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valphalowerrank

Proposition spine.3. *For any ordinal* α, $V_\alpha = \{x : \operatorname{rank}(x) \in \alpha\}$.

Proof. If $\operatorname{rank}(x) \in \alpha$ then $x \subseteq V_{\operatorname{rank}(x)} \in V_\alpha$, so $x \in V_\alpha$ as V_α is potent (invoking **??** multiple times). Conversely, if $x \in V_\alpha$ then $x \subseteq V_\alpha$, so $\operatorname{rank}(x) \leq \alpha$; now a simple transfinite induction shows that $x \notin V_\alpha$. □

Problem spine.2. Complete the simple transfinite induction mentioned in Proposition spine.3.

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rankmemberslower

Proposition spine.4. *If* $B \in A$, *then* $\operatorname{rank}(B) \in \operatorname{rank}(A)$.

Proof. $A \subseteq V_{\operatorname{rank}(A)} = \{x : \operatorname{rank}(x) \in \operatorname{rank}(A)\}$ by Proposition spine.3. □

Using this fact, we can establish a result which allows us to prove things about *all sets* by a form of induction:

Theorem spine.5 ($\in$-Induction Scheme). *For any formula* φ*:*

$$\forall A((\forall x \in A)\varphi(x) \to \varphi(A)) \to \forall A \varphi(A).$$

Proof. We will prove the contrapositive. So, suppose $\neg\forall A\varphi(A)$. By Transfinite Induction (**??**), there is some non-φ of least possible rank; i.e. some A such that $\neg\varphi(A)$ and $\forall x(\operatorname{rank}(x) \in \operatorname{rank}(A) \to \varphi(x))$. Now if $x \in A$ then $\operatorname{rank}(x) \in \operatorname{rank}(A)$, by Proposition spine.4, so that $\varphi(x)$; i.e. $(\forall x \in A)\varphi(x) \wedge \neg\varphi(A)$. □

Here is an informal way to gloss this powerful result. Say that φ is *hereditary* iff whenever every element of a set is φ, the set itself is φ. Then $\in$-Induction tells you the following: if φ is hereditary, every set is φ.

To wrap up the discussion of ranks (for now), we'll prove a few claims which we have foreshadowed a few times.

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ranksupstrict

Proposition spine.6. $\operatorname{rank}(A) = \operatorname{lsub}_{x \in A} \operatorname{rank}(x)$.

Proof. Let $\alpha = \text{lsub}_{x \in A} \text{rank}(x)$. By Proposition spine.4, $\alpha \leq \text{rank}(A)$. But if $x \in A$ then $\text{rank}(x) \in \alpha$, so that $x \in V_\alpha$ by Proposition spine.3, and hence $A \subseteq V_\alpha$, i.e., $\text{rank}(A) \leq \alpha$. Hence $\text{rank}(A) = \alpha$. □

Corollary spine.7. *For any ordinal α, $\text{rank}(\alpha) = \alpha$.* sth:spine:rank:ordsetrankalpha

Proof. Suppose for transfinite induction that $\text{rank}(\beta) = \beta$ for all $\beta \in \alpha$. Now $\text{rank}(\alpha) = \text{lsub}_{\beta \in \alpha} \text{rank}(\beta) = \text{lsub}_{\beta \in \alpha} \beta = \alpha$ by Proposition spine.6. □

Finally, here is a quick proof of the result promised at the end of **??**, that $\mathbf{ZF}^-$ proves the conditional *Regularity* ⇒ *Foundation*. (Note that the notion of "rank" and Proposition spine.4 are available for use in this proof since—as mentioned at the start of this section—they can be presented using $\mathbf{ZF}^-$ + Regularity.)

Proposition spine.8 (working in $\mathbf{ZF}^-$ + Regularity). *Foundation holds.* sth:spine:rank:zfminusregularityfoundation

Proof. Fix $A \neq \emptyset$, and some $B \in A$ of least possible rank. If $c \in B$ then $\text{rank}(c) \in \text{rank}(B)$ by Proposition spine.4, so that $c \notin A$ by choice of B. □

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Bibliography